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ALEXANDER GROTHENDIECK'S WORK ON FUNCTIONAL ANALYSIS

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Abstract. Alexander Grothendieck obtained the Medal Fields in 1966 for his contributions to Homological Algebra and Algebraic Geometry. However, Grothendieck's work on Functional Analysis, appeared in 25 papers between 1950 and 1957, had a tremendous and deep influence in the development of this area of Mathematics. Along this paper, we shall try to give a perspective of this work, Grothendieck's ideas to study and introduce new properties in topological vector spaces and a quick look at part of his heritage.

Dedicated to the Memory of Miguel de Guzmán.

Introduction.

Alexander Grothendieck is one of the most influential mathematicians of the twentieth century. He received the **Field's** Medal in 1966 *"for his contributions to Homological Algebra and Algebraic Geometry"*, but this is to say little about the impact of Grothendieck's work in modern Mathematics. Quoting [Ca1]:

> *The mere enumeration of Grothendieck's best known contributions is overwhelming: topological tensor products and nuclear spaces, sheaf cohomology as derived functors, schemes, K-theory and Grothendieck-Riemann-Roch, the emphasis on working relative to a base, defining and constructing geometric objects via the functors they are to represent, fibred categories and descent, stacks, Grothendieck topologies and topoi, derived categories, formalisms of local and global duality, étale cohomology and the cohomological interpretation of L-functions, crystalline cohomology, "standard conjectures", motives and the "yoga of*

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> *weights", tensor categories and motivic Galois groups. It is difficult to imagine that they all aprang from a single mind.*

The above cite is impressive, and surely part of the theories mentioned are unfamiliar to most of the readers. And it is not less impressive to know that the list of "official" Grothendieck's publications begins in 1950 (with a paper appeared in *C. R. Acad. Sci Paris*), and ends in 1974 (*Groupes de Barsottti-Tate et cristaux de Dieudonné*, Séminaire de Mathématiques Supérieures. **45** (Été 1970). Les Presses d l'Université de Montréal, 1974)[1].

Several colleagues and friends refers his total dedication to the research, living alone and working for 25 or 26 hours each "day". Hence, why did he abruptly ended a career so fertile at the age of 42? The official reason given by Grothendieck himself to resigned his position at the *Intitut des Hautes Etudes Scientifiques (IHES)* was that he had discovered that the Ministry of Defense had partly subsidized the Institute. But it seems that the reaction is, at least, exaggerated. **P. Cartier**, friend and colleague at the IHES, refers in [Ca2] his opinions about this fact:

> *...He* ([Grothendieck]) *is the son of a militant anarchist who had devoted his life to revolution... He lived as an outcast throughout his entire childhood and was a "displaced person" for many years, traveling with a United Nations passport* ([is citizenship papers disappeared in Berlin, during 1945]*... He lived his principles, and his home was always wide open to "stray cats". In the end, he came to consider Bures-sur-Yvette* [where was located the IHES] *a gilded cage that kept him away from real life. To this reason, he added a failure of nerve, a doubt as to the value of scientific activity... He confided his doubts to me and told me that he was considering activities other than mathematics. One should perhaps add the effect of a well-known "Nobel syndrome"... yielding to the pernicious view that sets 40 as the age when mathematical creativity ceases. He may have believed that he had passed his peak and that thenceforth he would be able only to repeat himself with less effectiveness.*
>
> *The mood of the time also had a strong influence. The disaster that had been the second Viet Nam war, from 1963 to 1972, has awakened many consciences... A significant number of French mathematicians took concrete action and traveled to Hanoi, as he (and I) did... The cold war was at its height, and the risk of a nuclear confrontation was very real. The problems of overpopulation, pollution, and uncontrolled development -everything that is now classified as ecology- had also begun to attract attention. There were plenty of reasons to call science into question!*

[1]A list of all Grothendieck's papers on Functional Analysis is included as Appendix at the end of the paper

I think that this long cite gives some light about Grothendieck's character and his personal circumstances. For the sake of completeness, probably it should be convenient to give a quick look at Grothendieck's timeline. But before proceeding, one remark: Most of the available biographical information about Grothendieck coincides in the fundamental, but there is a constant reference to the inaccuracies of the *other* biographies. I have followed essentially the biographical notes included in [Ca2] and the information appeared at the web page `http://www.math.jussieu.fr/~leila/biog.html` entitled *a brief timeline for the life of Alexander Grothendieck* (with the added subtitle: *which has the advange of accuracy*):

Alexander Grothendieck was born in Berlin, in 1928. His father, **Alexander Schapiro** (called Sascha) was a revolutionary Jew born in Russia, who participated in many of the revolutionary movements occurred in central Europe during the first decades of the twentieth century. In the 1920's he lived in Germany, working as a street photographer and fighting politically against Hitler and the Nazis. He met there **Johanna Grothendieck** (called Hanka) a german jewish woman sharing Sascha's ideals. After 1933, the couple fled to Paris, leaving young Alexander with a foster family in Hamburg. Sascha and Hanka joined the anarchists of the F.A.I. when the Spanish Civil War broke out in 1936. They returned to France in 1939 and Hanka found work in Nîmes.

Alexander's foster mother decided to send him to joint his parents in France in 1939, due to the political situation. After the French defeat in 1940, Vichy's collaborationist government promulgated anti-jewish laws for the unoccupied zone, and Hanka and his son Alexander were interned in the Rieucros Camp. Sascha was interned in Le Vernet camp and then sent directly to Auschwitz, where he died in 1942.

Also in 1942 the Rieucros camp is dissolved. Alexander was sent to the village Le Chambon sur Lignon and was housed at the Swiss Foyer, attending the *Collège Cévenol* during his baccalaureat. He studied mathematics at the University of Montpellier from 1945 to 1948 and, having finished his licensure, he went to Paris for his doctorate, with a letter of recommendation to **Élie Cartan**. His son, **Henri Cartan** advices Grothendick to do his doctorate in Nancy with **Laurent Schwartz**. He finished his Thesis (a real masterpiece) in 1953 and spent the following two years in Brazil. In 1955 he visited the Universities of Kansas and Chicago and hoped to find a position in France, in spite of his foreign nationality.

In 1959, the IHES is created in Bures-sur-Yvette and a research position is offered to Grothendieck. During the 12 years spended there, he renewed completely the Algebraic Geometry. His *Élements de Géométrie Algébrique* (he wrote, with Dieudonné, 4 volumes) and the series *Séminaire de Géométrie Algébrique* (one per year, from 1960-61 to 1967-68; some of them divided in several volumes) form an epoch-making contribution.

As we have said, Grothendieck left the IHES in 1970. After then, he became vividly interested in Ecology for some time, and founded the group

Survivre et Vivre. For two years, he had a temporary post at the prestigious *Collége de France* in Paris, but he used his lectures to talk more about questions of ecology and peace than of mathematics.

In 1972 Grothendieck obtains the French nationality and from 1973 to 1984 he lectures at Montpellier University. In 1984 he applied for a position of Director or Research at CNRS, specifying that he did not want to have any regular research duty. After some long discussion at the National Committee, the position was given to Grothendieck.

In 1988 he retired officially. At the same year he was awarded (jointly with his student **Pierre Deligne** the Crafoord Prize, from the Swedish Royal Academy of Sciences, but he declined the prize on ethical grounds. A letter explaining his reasons, appeared in 1989 (see [G7]).

In 1991 he left his home suddenly and disappeared. He is said to live in some part of the Pyrenées, refusing practically every human contact and spending his time dedicated to meditation on philosophical questions. The monumental manuscript *Récoltes et Semailles* ("Harvesting and Sowing"), a kind of very personal autobiography, with more than 2000 pages, is now available on Internet in several languages.

There is an interesting web site devoted to Grothendieck, `http://www.grothendieck-circle.org/`, with a large amount of mathematical and biographical information, photos and links to other related sites.

Grothendieck and Functional Analysis

Grothendieck's contributions to Algebraic Geometry are well known for general mathematicians. A great part of the three volumes in [Ca1], published on the occasion of Grothendieck's sixtieth birthday, are devoted to these aspects of his work. [AJ] is also a good survey (in Spanish) on the work of Grothendieck, focusing mainly on those topics related to Algebraic Geometry. In the following pages, I shall try to give account of some of his contributions done at the beginning of his career: those related with Functional Analysis. They are included in 24 papers and one book, appeared between 1950 and 1957 (except a short note about the trace of certain operators between Banach spaces, published in 1961). Among them, there are some of the most influential works in the development of Functional Analysis.

I shall focus mainly my attention on his contributions to the theory of topological tensor products, first in the setting of locally convex spaces (the subject of his Thesis) and later as a powerful tool to study the structure of Banach spaces (mainly contained in the famous São Paolo's *Résumé*). Between these two great masterpieces, I shall briefly comment the remarkable paper on $\mathcal{C}(K)$-spaces appeared in the *Canadian J. Math.*, where Grothendieck axiomatizes some important properties. And not only for its transcendence in the future development of the theory, but because I think is a paradigmatic example of Grothendieck's way of thinking. In any case, I'll try not to be very technic, giving always priority to clearness over precision.

The Thesis: "Produits tensoriels topologiques et espaces nucléaires."

As we have seen, Grothendieck went to Paris to do his doctorate in 1949. After attending some courses there, he followed the advise of H. Cartan and went to Nancy to work with **Laurent Schwartz** and **Jean Dieudonné**. Let Schwartz himself describe his impressions:

> *We* [Dieudonné and Schwartz] *received Grothendieck in october, 1951. He showed to Dieudonné a 50 pages paper on "Integration with values in a topological group". It was exact, but with no interest at all. Dieudonné, with all the aggressiveness he could have,* ([and he could a lot] *gave him a severe ticking-off, arguing that he should not work in such a manner, just generalizing for the pleasure of doing so...Dieudonné was right, but Grothendieck never admitted it...*
>
> *We had just published a paper on "Les espaces $\mathcal{F}$ et $\mathcal{LF}$"... that included 14 questions, problems that we were not able to solve. Dieudonné proposed Grothendieck to think about some of them, those that he preferred. We did not see him for some weeks. When he appeared again, he had solved one half of the questions! Deep and difficult solutions which needed new notions. We were wondered.* [Sch, p. 292-293].

Schwartz realized at once that he had met a mathematician of first order, and in the spring of 1953 he proposed to Grothendieck, as a subject of his Thesis, the general problem of finding a "good" topology on the tensor product $E \bigotimes F$ of two locally convex spaces E and F. At this time, Schwartz was starting the theory of vector-valued distributions, that is, the study of the space $\mathcal{D}'(F) := \mathcal{L}(\mathcal{D}, F)$ of continuous linear operators from the test space $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ (the space of C^∞ scalar functions with compact support, endowed with his usual inductive limit topology) into the locally convex space F. A good topology for this space was evident, inducing a corresponding good topology on its dense subspace $\mathcal{D}' \otimes F$. But this was not obvious in general.

Grothendieck spent the summer of 1953 in Brazil and, at the end of July, he wrote to Schwartz a, in certain sense, deceptive letter: on $E \otimes F$ there were two topologies, as natural one as the other, and different! Schwartz did not know what to say, because on $\mathcal{D}' \otimes F$ there were *one* natural topology. Fortunately, two weeks latter a triumphal letter arrived: the two natural topologies coincide on $\mathcal{D}' \otimes F$ (and with the natural one)!

The two natural topologies discovered for Grothendieck are what we now know as the *projective* or π-topology and the *injective* or ϵ-topology. The π-topology is the greatest "reasonable" one (in some precise sense), and makes, with the natural identifications, $\mathcal{L}(E \otimes_\pi F, G) = \mathcal{B}(E \times F, G)$, all the continuous bilinear maps from $E \times F$ into G. The ϵ-topology is the least "reasonable" topology and had a great importance in the development of important classes of operators.

Grothendieck's Thesis was completed in 1953. It is a masterpiece of more than 300 pages which contains not only the main theorems of the theory of topological tensor products, but also new methods, technics and a lot of seminal ideas which were to renew Functional Analysis. But let us hear the opinion of his Thesis' advisor, L. Schwartz:

> *... "It is a monument, a masterpiece of the first order. It was necessary to read it, to understand it, to learn from it, because it was difficult and deep. It took to me six months at full time. What a hard work, but what a joy!... I learnt a lot of new things. It was the most beautiful of* "my" *Thesis..."* [Sch, p. 294]

Besides the definition of the π and ϵ topologies on a tensor product of locally convex spaces and a deep study of this new objects, examples and applications to the study of vector-valued function spaces, etc., the Thesis contains much more. We shall mention some of the contents:

I.- *The Approximation Property.* Since the π-topology is finer than the ϵ-topology, we can extend the identity operator to the respective completions, obtaining a canonical map

$$E\widehat{\otimes}_\pi F \to E\widehat{\otimes}_\epsilon F,$$

not necessarily injective, as Grothendieck points out. He says that he does not know any example where the injectivity fails, and poses the *Problème de biunivocité* and its relative, the *Problème d'approximation.* He gives a great number of different formulations of these problems and introduces a sufficient condition to answer in the positive both questions: E or F to have the *condition d'approximation* (approximation property), which means that the identity operator can be uniformly approximate on precompact subsets by operators of finite rank. When considering Banach spaces, he introduces also the *metric approximation property* (just imposing that the approximating operators have norm ≤ 1). He proves that the classical Banach spaces, their duals, biduals, etc. have the metric approximation property. Also the nuclear spaces and most of the usual locally spaces that appear in the Theory of Distributions have the approximation property. Then Grothendieck states as an important problem to know if *every* locally convex space (equivalently, every Banach space) has the approximation property. He gives several equivalent conditions and different properties of permanence, showing in particular that it is equivalent to the problem No. 153 of the famous *Scottish Book*, created by **S. Banach** in 1935 to write down the problems posed by the group of mathematicians joined around Banach and Steinhaus in Lwow and their invited visitors. Problem 153 was posed in November 6, 1936 by **S. Mazur**, and the prize offered for the solution was a live goose. And, in fact, Mazur had the opportunity of giving the prize in 1973 to **Per Enflo**, who had solved in the negative the conjecture the year before. Of course, Grothendieck's work was fundamental in the later work on the problem and its negative solution obtained.

II.- *Nuclear, integral and related operators.* When E, F are Banach spaces, the natural inclusion $E' \bigotimes F \hookrightarrow \mathcal{L}(E,F)$ can be extended (being $\mathcal{L}(E,F)$ complete with its usual norm) to a map $E\widehat{\otimes}_\pi F \to L(E,F)$. The operators in the image of this map are called by Grothendieck *nuclear*[] *operators*. In the general case, an operator $T : E \to F$ between locally convex spaces is called *nuclear* if it can be factorized in the form $T = A \circ S \circ B$, with S a nuclear operator between two Banach spaces. Grothendieck fulfills a deep study of this class of operators, and gives many examples.

Another important class of operators isolated by Grothendieck is that of *integral operators:* Since the ϵ-topology is coarser that the π one, the topological dual $E\widehat{\otimes}_\epsilon F)'$ is a subset $J(E,F) \subset \mathcal{B}(E,F)(\equiv (E\widehat{\otimes}_\pi F)')$. The bilinear forms in $J(E,F)$ are called *integral* by Grothendieck, who gives also a characterization in terms of an integral representation. A linear operator $T : E \to F$ is *integral* if the corresponding bilinear form on $E \times F'$ is integral. In case of Banach spaces, Grothendieck gives immediately a factorization criterium: $T : E \to F$ is integral if and only if the composition of T with the canonical embedding of F in its bidual F" can be factorized in the form $E \to L_\infty(\mu) \xrightarrow{i} L_1(\mu) \to F$",where μ is a probability on some compact space and i is the natural inclusion.

This method of factoring an operator through classical Banach spaces in order to take advantage of the knowledge of these spaces, probably was not due to Grothendieck, but he made a systematic use of it, and this is one of the great heritage that Functional Analysis received from him.

Grothendieck carries out a deep study of integral maps, with surprising applications to classical analysis, summable sequences, vector measures, etc.

The nuclear maps are always compact and integral, and the composition of two integral maps is nuclear.

In the case of Banach spaces, Grothendieck introduced also, other interesting classes of operators: the *right* (resp.,*left*) *semi-integral* (or "pre-integral") operators $T : E \to F$, just imposing that the composition with an embedding of F into a L_∞-space (resp., with a quotient map from an L_1-space onto E) be integral. The right semi-integral operators are precisely the familiar *absolutely summing operators*, generalized and studied in the sixties by **Mityagin, Pelczynski** and **Pietsch**, among others, to the important class of *absolutely p-summing operators* $(1 \leq p < \infty)$.

III. *The kernel theorem and Nuclear spaces.* During the International Congress of Mathematics of 1950, Schwartz had announced his surprising *Théorème des noyaux* ("Kernel Theorem"), asserting that *every* continuous linear operator $T \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ $(\equiv \mathcal{B}(\mathcal{D} \times \mathcal{D})$, i.e., the continuous bilinear maps on $\mathcal{D} \times \mathcal{D})$, came from a "distributional kernel", that is, a distribution in two variables $K(x,y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^m)$ such that for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^m)$,

$$T(\varphi)(\psi) = \langle \varphi(x)\psi(y), K(x,y)\rangle = \langle \varphi \otimes \psi, K\rangle$$

that could be written formally as $\int K(x,y)\varphi(x)\psi(y)\, dxdy$. This is really surprising, because it is not true for most of the usual function spaces. For instance, the identity operator in the usual L_2 space, cannot be expressed as a kernel operator.

In general, since the pioneering works of **D. Hilbert** and **F. Riesz** it was well known that the operators on some function space, like L_2, of the type

$$T(f)(x) := \int K(x,y)f(y)\,dy$$

(*kernel operator*) had especially good properties, but unfortunately they did not exhaust all the possible operators. And, by the way, this was one of the main difficulties in the rigorous formulation of Quantum Mechanic. The idea of **P. Dirac** and others was to express any *observable* ($\sim$ "linear operator") in terms of a base formed by the "states" (ψ_p) of the system (the eigenfunctions corresponding to the eigenvalues of the operator, that is, the "spectrum" of the observable). But in most cases, this spectrum was not countable (in words of Dirac: *"... the total number of independent states is infinite, and equal to the number of points of a line"* [DIR]) Then the corresponding "matrix" (α_{pq}) was indexed by two continuous parameters, $p, q \in \mathbb{R}$, and the action of the observable α on some state ψ_q should be written in the form

$$\alpha(\psi_q) = \int_{\mathbb{R}} \alpha_{pq}\psi_p\,dp,$$

that is, as a kernel operator. But then, if one wants to represent in this way the operator "multiplication for a non zero constant c", we need a kernel of the type $\alpha_{pq} = c\delta(p-q)$, that is, a *singular function.* Dirac used also other singular functions and its derivatives, just applying formally the method of "integration by parts".

Hilbert tried to follow a similar method, but the appearance of singular functions, made him to look for another point of view: this was the **J. von Neumann**'s spectral theory of (non necessarily bounded) operators on subspaces of a Hilbert space.

The kernel theorem allows to justify rigorously part of the ideas of Dirac: in fact, if E and F are function spaces over open subsets of $U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^m$ usually we have

$$\mathcal{D}(U) \hookrightarrow E \hookrightarrow \mathcal{D}'(U)$$

and

$$\mathcal{D}(V) \hookrightarrow F \hookrightarrow \mathcal{D}'(V)$$

(with continuous embeddings, the first one with dense range). Hence, every continuous operator $S : E \to F$ gives rise, by composition, to a continuous operator $T : \mathcal{D}(U) \to \mathcal{D}'(V)$. Therefore, it can be represented by a (distributional) kernel operator!

This long detour tries to explain why a great part of Grothendieck's thesis is devoted to the study and characterization of those locally convex spaces E such that $E \otimes_\pi F = E \otimes_\epsilon F$ for every locally convex space F. He called such spaces *nuclear spaces.* The reason is that the spaces $\mathcal{D}$ (and $\mathcal{D}'$) are nuclear, and the kernel theorem is a trivial consequence of this fact and the (easy) result that $\mathcal{D}(U) \otimes_\epsilon \mathcal{D}(V)$ is a *dense* topological subspace of $\mathcal{D}(U \times V)$: In fact, let $T : \mathcal{D}(U) \to \mathcal{D}'(V)$ be a continuous linear map, and let $B : \mathcal{D}(U) \times \mathcal{D}(V) \to \mathbb{K}$ be the corresponding continuous bilinear map. Then,

$$B \in \mathcal{B}(\mathcal{D}(U) \times \mathcal{D}(V), \mathbb{K}) = (\mathcal{D}(U) \widehat{\otimes}_\pi \mathcal{D}(V))' = (\mathcal{D}(U) \widehat{\otimes}_\epsilon \mathcal{D}(V))' = \mathcal{D}'(U \times V)$$

(where the $\widehat{}$ means the completion of the corresponding space), which is essentially the content of the kernel theorem, *via* the natural identifications.

Grothendieck carries out a deep study of these class of locally convex spaces (which contains no Banach space of infinite dimension), proving that they enjoy very good properties of stability and permanence, and giving many examples and applications.

By the way, this is a typical example of Grothendieck's way of doing mathematics: put the problem in a more general setting and find a general theory (usually, very deep and far-reaching) which contains the solution of the initial problem as a particular case. Of course, this is the way as most of the twentieth century mathematics were developed, but usually the general theories were created by many authors along a certain time. In Grothendieck's work, this is a constant procedure!

Grothendieck's Thesis [G1] contains, of course, much more. I have just tried to give a quick look at it, mentioning some of the most relevant results presented there. It also contains a great number of open questions and problems that motivated a great research activity when Grothendieck's work became to be known among the specialists. The Thesis appeared published in 1955, as the Vol. No. 16 of the prestigious *Memoir of the American Mathematical Society.* Since it took such a long time to be published, Grothendieck wrote a survey ([G2]) quoting some of the more relevant results and as available references for his later works on the subject.

The Dunford-Pettis and relatives properties

In 1953 appeared in the Canadian Journal of Mathematics the paper [G3], *"...devoted essentially to the study of the weakly compact linear operators from a $\mathcal{C}(K)$-space into an arbitrary locally convex space F.* ([G3], introduction). By transposition, this is equivalent to the study of weakly compact subsets of the space of Radon measures on K (the dual of $\mathcal{C}(K)$). And, in fact, the paper contains some of the most useful weak compactness criteria on the spaces of Radon measures. But it contains much more.

Grothendieck's favorite method of studying general classes of operators by factoring them through "classical" spaces of type $\mathcal{C}(K)$, $L_1(\mu)$ or Hilbert spaces (used in his *Thesis*, but much more in the forthcoming *Résumé*), made quite important to know the behavior of different classes of operators on these spaces. And there are in the paper several important results in this direction, starting with a Riesz-type representation in terms of vector measures. He remarked that, from the Riesz's classical representation theorem, any operator $T : C(K) \to E$ could be represented in the form

$$T(f) = \int_K f\,dm$$

where m is a regular, finitely additive vector measure with finite semi-variation on the Borel subsets of K, with values in E" (the *representing measure* of T). Grothendieck proved that T is weakly compact if and only if its representing measure takes values in E or, equivalently, it is countably additive.

The relationships between properties of the operator and its representing vector measure will be widely used in the later work on this subject, being very fruitful for both theories: linear operators on $\mathcal{C}(K)$ spaces (and also on vector valued function spaces $\mathcal{C}(K, E)$) and vector measures.

On the other hand, the article emphasizes the "functorial" point of view of Grothendieck: In order to study the structure of some mathematical object, you have to look at the behavior of the morphisms on and into it. This was quite usual in some parts of Mathematics, but not so in Analysis. The paper contains the first systematic treatment of what I called in [Bo] the *homological method* for defining properties on Banach spaces (Grothendieck treats the general case of operators between locally convex spaces, but he also mention that *Ce travail pu se traiter sans sortir du cadre des espaces de Banach.*).

The general scheme, as exposed in [Bo], is the following: Let Θ, Φ be two classes of linear operators between Banach spaces (in such a way that $\Theta(E, F)$, $\Phi(E, F)$ denote subsets of $\mathcal{L}(E, F)$ for every pair E, F of Banach spaces), and let $\mathcal{E}$ be a certain class of Banach spaces. We shall say that E *has property* $P(\Theta, \Phi; \mathcal{E})$, and we'll write $E \in P(\Theta, \Phi; \mathcal{E})$, if

$$\Theta(E, F) \subset \Phi(E, F), \quad \text{for every } F \in \mathcal{E}.$$

(When $\mathcal{E}$ is the class of *all* Banach spaces, we shall omit its mention, writing simply $P(\Theta, \Phi)$). Clearly, the property consider could be interested only when the defining relation does not hold trivially.

Usually, the classes Θ and Φ have some structure; more concretely, they are usually *operator ideals* (which means that the class is stable under composition with continuous linear operators and $\Theta(E, F)$ is a vector subspace of $\mathcal{L}(E, F)$, containing the finite rank operators). In this case, property $P(\Theta, \Phi; \mathcal{E})$ is an isomorphic invariant, stable under finite products and passing to complemented subspaces. It is easy to prove that when Φ is a *surjective* operator ideal, $P(\Theta, \Phi; \mathcal{E})$ is also stable under the formation of quotients by closed subspaces.

In order to give some examples, let us consider the following classes of operators:

-$\mathcal{L}$: all the operators.

-$\mathcal{K}$: the *compact* operators, i.e., those sending bounded set into relatively compact subsets.

-$\mathcal{W}$: the *weakly compact* operators, i.e., those sending bounded sets into *weakly* relatively compact subset.)

-$\mathcal{DP}$: the *Dunford-Pettis* (or *completely continuous*) operators, i.e., those that sends weakly convergent sequences into norm convergent ones (equivalently, they transform weakly compact subsets into norm compact subsets.)

-$\mathcal{D}$: the *Dieudonné* (or *weakly completely continuous*) operators, i.e., those which transforms weakly Cauchy sequences into weakly convergent ones.

All the above classes are operators ideals, closed under the usual operator norm. Besides, $\mathcal{K}$ and $\mathcal{W}$ are surjective. It is also clear that

$$\mathcal{K} \subset \mathcal{W} \subset \mathcal{D} \subset \mathcal{L} \tag{1}$$

and

$$\mathcal{K} \subset \mathcal{DP} \subset \mathcal{D} \subset \mathcal{L}, \tag{2}$$

with strict inclusions, and with no other general relation. With our notations, we have obviously:

- $E \in P(\mathcal{L}, \mathcal{K})$ if and only if E is finite dimensional.

-$E \in P(\mathcal{L}, \mathcal{W})$ if and only if E is reflexive.

-$E \in P(\mathcal{L}, \mathcal{DP})$ if and only if weakly convergent sequences in E are norm convergent, i.e., E is a *Schur* space.

-$E \in P(\mathcal{L}, \mathcal{D})$ if and only if E is weakly sequentially complete.

The Dunford-Pettis property.

This was the first property introduced by Grothendieck in the paper we are considering. His motivation was a long article by **N. Dunford** and **J. Pettis** appeared in 1940, in which they proved that weakly compact operators on L_1-spaces were (in our notation) Dunford-Pettis operators (i.e., $L_1 \in P(\mathcal{W}, \mathcal{DP})$). Important consequences were derived from this fact. Grothendieck axiomatized this property and called it the *Dunford-Pettis Property* (DPP in short). He gave several equivalent formulations and proved immediately than the property pass from E' to E. Since the dual of an L_1 space (built over a Radon measure) is an L_∞ space, hence isomorphic to a $\mathcal{C}(K)$ space, it is enough to prove that this last space enjoys the DPP for recovering the Dunford and Pettis' result. And this is one of the important results contained in Grothendieck's memory.

The DPP is "far" from reflexivity, since reflexive spaces with the DPP are finite dimensional. Also, the DPP can be localized, in the sense that it coincides with the property $P(\mathcal{W}, \mathcal{DP}; \{c_0\})$.

The DPP has been extensively studied (we remit the interested reader to the survey [Di]), and it gives important information on the structure of the spaces having it . Besides the $\mathcal{C}(K)$ and L_1-spaces, the disc algebra A (the space of all continuous functions on the unit disc $D := \{z \in \mathbb{C} : |z| \leq 1\}$, which are analytic in the open unit disc), their analogous d-dimensional, the ball algebra $A(B_d)$ and the polydisc algebra $A(D^d)$, and the space H^∞ of all bounded analytic functions on the open unit disc, enjoy the DPP (the last three results are due to **J. Bourgain**, in a series of deep papers published in 1983-84 in *Studia Math.* and in *Acta Math.*. He also gave a correct proof of a result announced by Grothendieck: the space $C^k(U)$ of all complex-valued functions which are continuous with all their derivatives or order $\leq k$, on a d-dimensional compact manifold U, has the DPP).

The Reciprocal Dunford Pettis and the Dieudonné properties.

Following the same idea, Grothendieck introduces two more properties. In our notation:

-$P(\mathcal{DP}, \mathcal{W})$: the *reciprocal Dunford-Pettis property* (RDPP in short; the reason of the name is obvious.)

-$P(\mathcal{D}, \mathcal{W})$: the *Dieudonné property* (DP in short; the name is due to a Dieudonné's result on weakly convergent sequences of Radon measures).

In reality, any of the above properties is equivalent, by duality, to a weak compactness criteria in the dual of the space enjoying it. And this is the way as Grothendieck proved that $\mathcal{C}(K)$-spaces enjoy both properties. He also proved that both properties were stable under complemented subspaces, finite products and quotients (a trivial consequence of the mentioned fact that $\mathcal{W}$ is a surjective operator ideal).

From the inclusions (1) and (2) it is clear that DP implies the RDPP. Also, a weakly sequentially complete space E (that is, $E \in P(\mathcal{L}, \mathcal{D})$) has the DP if and only if $\mathcal{L}(E, \cdot) = \mathcal{W}(E, \cdot)$, i.e., E is reflexive. Consequently, no infinite dimensional L_1-space enjoys the DP. And, since such a space contains a complemented copy of ℓ_1, it also fails the RDPP. On the other hand, Rosenthal's dichotomy theorem yields immediately that If E contains no copy of ℓ_1, it enjoys the DP and the RDPP.

Grothendieck gave several applications of his results to the structure of classical Banach spaces.

The "Grothendieck spaces" and the hereditary properties.

Last chapter in Grothendieck's paper is devoted to some particular classes of $\mathcal{C}(K)$ spaces. In the first part he considers the case when K is a *stonean space* (or *extremally disconnected*), what means that the closure of every open set is open. This is equivalent to the fact that $\mathcal{C}_{\mathbb{R}}(K)$ is a complete lattice for the usual pointwise order. A typical example is the Stone-Cech compactification of any discrete topological space. Every $L_\infty(\mu)$-space is isomorphic to a $\mathcal{C}(K)$-space with K stonean.

Grothendieck proved that *any* continuous linear operator from such a $\mathcal{C}(K)$-space into a *separable* Banach space, is weakly compact. In other words, if $\mathcal{S}$ denotes the class of all separable Banach spaces, the spaces $\mathcal{C}(K)$ with K stonean verify property $P(\mathcal{L}, \mathcal{W}; \mathcal{S})$. These spaces are now known as *Grothendieck spaces.* An internal characterization (obtained also by Grothendieck) is that weak* convergent sequences in the dual space are weakly convergent. Obviously, reflexive spaces are Grothendieck (and they are the only separable Grothendieck spaces). Let us add that **J. Bourgain** proved in 1983 that H^∞ is a Grothendieck spaces, and that the property can also be localized: it coincides with the $P)\mathcal{L}, \mathcal{W}; \{c_0\})$.

The last important result included in the paper asserts that every subspace of c_0 enjoys the DPP, the DP and the RDPP. This is the prototype of the so called (by obvious reasons) "hereditary properties". A crucial lemma for the proof is that every normalized weakly null sequence in c_0 contains a basic sequence equivalent to the usual c_0-basis.

The seminal ideas contained in this paper were not well appreciated for Banach space researchers for more than 10 years. But then, they became tremendously influential in the development of the theory.

As for the "homological method", let us mention that $E \in P(\mathcal{W}, \mathcal{K})$ is equivalent to E' being Schur and, by a result of Odell and Rosenthal, $E \in P(\mathcal{DP}, \mathcal{K})$ if and only if contains no copy of ℓ_1. Of course, when considering new classes of operators one obtains new properties (see [Bo].)

The São Paulo's "Résumé"

Surely many specialists in Banach spaces will subscribe the opinion of **A. Pietsch**, that this is *the most spectacular paper of modern Banach space theory* (and one of the most influential, I would add). It was submitted to the Bulletin of the Sao Paolo's Mathematical Society in June 1954, and it appeared in 1956, but was not grasped for more than 10 years. In 1968 appeared in *Studia Mathematica* a long paper of more than 50 pages ([LP]) intending to show to the mathematical community some of the jewels hidden in the *Résumé*. The authors wrote in the introduction:

> *"The main purpose of the present paper is to give a new presentation as well as new applications of the results contained in Grothendieck's paper...*
>
> *Though the theory of tensor products constructed in Grothendieck's paper has its intrinsic beauty we feel that the results of Grothendieck and their corollaries can be more clearly presented without the use of tensor products....*
>
> *The paper of Grothendieck is quite hard to read and its results are not generally known even to experts in Banach space theory..."*

And, in fact, the authors bypassed the language of tensor products, by using systematically what now is konwn as *p-summing operators*, whose foundation had appeared in another seminal paper of **A. Pietsch** published also in *Studia* in 1967.

But let us come back to Grothendieck's *Résumé*. The underlying idea in the paper is to obtain new classes of operators between Banach spaces by defining suitable norms, on $E \otimes F$. When E and F are finite-dimensional, $E' \otimes F = \mathcal{L}(E, F)$, and a norm in $E' \otimes F$ defines an operator norm. The extension of this procedure to infinite-dimensional spaces is not trivial. It involves a skillful use of the so called *trace duality* (in the finite dimensional case, this essentially means the duality between the spaces $E \otimes F \equiv L(F^*, E)$

and $(E \otimes F)^* \equiv L(E, F^*)$, given by the trace of the composition). Obviously, this cannot be trivially extended to the setting of infinite-dimensional Banach spaces and continuous linear maps. Grothendieck was aware of this problem and he devoted the firs chapter of the Memory to establish a method for defining "good" norms on a tensor product of Banach spaces: the $\otimes$-norms (or *tensor norms*). Such a norm $\|\cdot\|_\alpha$ should be, in the firs place, *reasonable*, what means that $\|x \otimes y\|_\alpha = \|x\|\|y\|$ and $x' \otimes y' \in (E \otimes_\alpha F)'$, with (dual) norm $\|x' \otimes y'\|_{\alpha'} = \|x'\| \, \|y'\|$ (hence, $E' \otimes F'$ is a subspace of $(E \otimes_\alpha F)'$. The dual norm α' induces then on $E' \otimes F'$ another reasonable norm.)

Of course, the norms ϵ and π are reasonable (and duals one of the other). In fact, a norm α defined for every pair of normed spaces is reasonable if and only if $\epsilon \leq \alpha \leq \pi$.

On the other hand, the $\otimes$-norms should verify a good functorial property: the so called *metric mapping property*. This means that whenever $u_i \in \mathcal{L}(E_i, F_i)$ $(i = 1, 2)$, then $u_i \otimes u_2 \in \mathcal{L}(E_1 \otimes_\alpha E_2, F_1 \otimes_\alpha F_2)$, with norm $\leq \|u_1\| \, \|u_2\|$.

Next, Grothendieck gives a method to construct $\otimes$-norms with good duality properties: First, he considers a $\otimes$-norm α defined on the class FIN of all the *finite-dimensional* Banach spaces. Then, he *extends* this norm to every pair of Banach spaces in the following way: If E and F are normed spaces, for $u \in E \otimes F$ we define

$$\|u\|_{\overrightarrow{\alpha}} := \inf\{\|u\|_\alpha : u \in M \otimes N\},$$

when M and N run over the finite dimensional subspaces of E and F, respectively. This is what now is known as the *finite hull procedure* for extending a tensor norm from the class FIN to the class NORM or all normed spaces. There is another standard procedure (the *cofinite hull*, essentially due to **H. P. Lotz**) to extend a tensor norm α on FIN to a tensor norm $\overleftarrow{\alpha}$ on NORM (see [DF], Ch. II), that was not considered by Grothendieck. And there is a good reason for that. In fact, if E and F have the approximation property, $E \otimes_{\overrightarrow{\alpha}} F = E \otimes_{\overleftarrow{\alpha}} F$, and the first space without the approximation property was discovered 20 years after Grothendieck's *Résumé*!

Grothendieck considers some operations with the $\otimes$-norms on FIN, which are extended to NORM by the finite hull procedure. In particular, when $M, N \in$ FIN, for every $\otimes$-norm α, $M \otimes N = (M' \otimes_\alpha N')'$ (algebraically). The dual norm induced on $M \otimes N$ by this identification identification is denoted by α' and called the *dual norm* of α. It is also a $\otimes$-norm and its extension $\overrightarrow{\alpha'}$ is called the dual tensor norm of $\overrightarrow{\alpha}$. Since $\alpha" = \alpha$ on FIN, the same relation holds for their extensions.

Now Grothendieck proceeds to define the class of operators and bilinear forms associated to a tensor norm α: Since $\alpha \leq \pi$, for every pair of Banach spaces E, F, the dual of the completion $E \widehat{\otimes}_\alpha F$ of $E \otimes_\alpha F$ can be identified to a subspace of $\mathcal{B}(E, F)$, the dual of $E \widehat{\otimes}_\pi F$. Grothendieck calls a bilinear form B on $E \times F$ of *type* α if it belongs to the dual of $E \widehat{\otimes}_{\alpha'} F$. Its norm in this dual is denoted by $\|B\|_\alpha$. Analogously, a linear map $u : E \to F$ is of *type* α if its canonically associated bilinear map on $E \times F'$ is of type α. $\|u\|_\alpha$ will denote, obviously, the α- norm of the bilinear form. The class of all linear maps of type α from E to F, endowed with the α-norm, will be denoted by $\mathcal{L}^\alpha(E, F)$, that is

$$\mathcal{L}^\alpha(E, F) := (E\widehat{\otimes}_{\alpha'}F')' \cap \mathcal{L}(E, F).$$

(Even more, $\mathcal{L}^\alpha$ is a (maximal) *normed operator ideal*, in the sense of the theory later developed by Pietsch, There is a one-to-one correspondence between maximal normed operator ideals and tensor norms, and this duality has shown to be extremely useful.See [DF] for details.)

Since $\pi' = \epsilon$ and $\epsilon' = \pi$, the linear maps of type π are precisely the *integral* maps, and those of type ϵ are *all* the continuous linear maps. Instead of defining different tensor norms and look at the corresponding classes of linear maps (what was done much later by different authors, who identified the tensor norms that produces the absolutely p-summing, p-integral, p-dominated or (p, q)-factorable operators, among many others), Grothendieck develops a general theory of tensor norms, defining new operations (the *right* and *left projective* and *injective* hull of a tensor norms, connected to factorization properties of the associated linear maps through classical Banach spaces of type $\mathcal{C} = C(K)$, $L = L_1(\mu)$ and $H =$ Hilbert space.

Grothendieck proves the fundamental result that the if one starts with the ϵ-norm and takes duals, transposed, right or left injective or projective hulls finitely many times, then one obtains, up to equivalence, only 14 different tensor norms (the *natural* tensor norms, and each of them gives rise to a class of operators characterized by a typical factorization. ([G4, p. 37], [DF, Chapter 27])

Chapter 3 is devoted to the study of tensor norms on Hilbert spaces. The so called *hilbertian tensor norm* is introduced by the property that the corresponding linear maps factorize through a Hilbert spaces (in [DF] is designed as w_2; in [DJT] is noted as γ_2). The relationships with other tensor norms and the different classes of operators that appear, are studied. In particular, canonical factorization results for mappings $\mathcal{C} \to H$ and $H \to L$ are obtained.

But the deepest and most and influential results appear in Chapter 4: Theorem 4.1, which Grothendieck calls *théorème fondamental de la théorie métrique des produits tensoriels* states that the identity operator on a Hilbert space is what Grothendieck calls "preintegral", and its preintegral norm is bounded by an universal constant K_G (*Grothendieck's constant*). Grothendieck gave several other formulations (in particular, that the tensor norms π and w_2 are equivalent on $\mathcal{C} \otimes \mathcal{C}$) and obtained relevant applications to factorization of operators, Harmonic Analysis, summable sequences, etc. One of the most important achievements in the mentioned Lindenstrauss and Pelczynski's celebrated paper of 1968 was to realize the importance of this theorem and its reformulation in the form of an inequality involving $n \times n$

matrices and Hilbert spaces: *Let* (a_{ij}) *be an* $n \times n$ *scalar matrix. Then for each Hilbert space* H,

$$\sup\left\{\left|\sum_{i,j=1}^{n} a_{ij}(x_i|y_j)\right| : \|x_i\|, \|y_j\| \leq 1\right\} \leq K_G \sup\left\{\left|\sum_{i,j=1}^{n} a_{ij}s_it_j\right| : |s_i|, |t_j| \leq 1\right\}.$$

This is why the theorem is now called *Grothendieck's inequality.* Their proof owes much to that of Grothendieck, but this formulation, allied with the theory of p-summing operators, allows to avoid much of the machinery of tensor products to present many of Grothendieck's ideas. Just in [LP] several important results on the structure of $C(K)$ and $\mathcal{L}_p$ spaces (some of them due to Grothendieck himself!) are obtained. (See [DJT], Chapter 3 for a modern exposition).

As in the case of his Thesis, Grothendieck published some of the results of the *Résumé* before its appearance ([G5] and [G6]).

Since 2002 a series of articles under the generic title of *the metric theory of tensor products (Grothendieck's résumé revisited*) are appearing in the southafrican Journal *Quaestiones Math.*. The authors are **J. Diestel, J. Fourie** and **J. Swat** and for the moment have been published 5 papers (the last one in the No. 4 of Vol. **26**, corresponding to 2003.)

Conclusion

As we have seen, Grothendieck's life is astonishing, in the personal and as a mathematician. His influence in the twentieth century mathematics is enormous. And not only because of his magnificent results, but for his attitude and special vision. His look for general theories and methods, and the relationships between different areas of mathematics, opened new fields and lines of research, sometimes developed many years after his contribution.

Appendix

Grothendieck's publications on Functional Analysis.

(1) *Sur le complétion du dual d'un espace vectoriel localement convexe.* C. R. Acad. Sci. Paris **230** (1950), 605-606.

(2) *Quelques résultats relatifs à la dualité dans les espaces* ($\mathcal{F}$). C. R. Acad. Sci. Paris **230** (1950), 1561-1563.

(3) *Critères généraux de compacité dans les espaces vectoriels localement convexes. Pathologie des espaces* ($\mathcal{LF}$). C. R. Acad. Sci. Paris **231** (1950), 940-941.

(4) *Quelques résultats sur les espaces vectoriels topologiques.* C. R. Acad. Sci. Paris **233** (1951), 839-841.

(5) *Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liée à cette notion.* C. R. cad. ci. Paris **233** (1951), 1556-1558.

(6) *Critères de compacité dans les espacies fonctionnels généraux.* Amer. J. of Math. **74** (1952), 168-186.

(7) *Sur les applications linéaires faiblement compactes d'espaces du type C(K).* Canadian J. Math. **5** (1953), 129-173.

(8) *Sur les espaces de solutions d'une classe générale d'équations aux dérivées partielles.* J. Analyse Math. **2** (1953), 243-280.

(9) *Sur certains espaces de fonctions holomorphes, I.* J. reine angew. Math. **192** (1953), 35-64.

(10) *Sur certains espaces de fonctions holomorphes, II.* J. reine angew. Math. **192** (1953), 77-95.

(11) *Quelques points de la théorie des produits tensoriels topologiques.* Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio 1954, 173-177. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954.

(12) *Espaces vectoriels topologiques.* Instituto de Matematica Pura e Aplicada, Universidade de São Paulo, 1954.

(13) *Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires.* Ann. Inst. Fourier **4** (1952), 73-112.

(14) *Sur certains sous-espaces vectoriels de* L^p. Canadian J. Math. **6** (1954), 158-160.

(15) *Résultats nouveaux dans la théorie des opérations linéaires, I.* C. R. Acad. Sci. Paris **239** (1954), 577-579.

(16) *Résultats nouveaux dans la théorie des opérations linéaires, II.* C. R. Aad. Sci.Paris **239** (1954), 607-609.

(17) *Sur les espaces* ($\mathcal{F}$) *et* ($\mathcal{DF}$). Summa Brazil. Math. **3** (1954), 57-123.

(18) *Produits tensoriels topologiques et espaces nucléaires.* Mem. Amer. Math. Soc. No. 16, 1955.

(19) *Une caractérisation vectorielle-métrique des espaces L^1.* Canad. J. Math. **7** (1955), 552-561.

(20) Erratum au mémoire *Produits tensoriels topologiques et espaces nucléaires.* Ann. Inst. Fourier **6** (1955-56), 117-120.

(21) *Résumé de la théorie métrique des produits tensoriels topologiques.* Bol. Soc. Mat. São Paulo **8** (1956), 1-79.

(22) *La théorie de Fredholm.* Bull. Soc. Math. France **84** (1956), 319-384.

(23) *Sur certaines classes de suites dans les espaces de Banach, et le théorème de Dvoretzky-Rogers.* Bol. Soc. Mat. São Paulo **8** (1956), 81-110.

(24) *Un résultat sur le dual d'une C^*-algèbre.* J. Mat. Pures Appl. **36** (1957), 97-108.

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[G2] A. Grothendieck, *Rèsumé des resultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires.* Ann. Inst. Fourier Grenoble **4** (1952), 73-112 (appeared in 1954).

[G3] A. Grothendiek, *Sur les applications linéaires faiblement compactes d'espaces du type C(K).* Canadian J. Math., **5** (1953), 129-173.

[G4] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques.* Bol. Soc. Mat. Sao Paulo **8** (1953), 1-79 (appeared in 1956).

[G5] A. Grothendieck, *Résultats nouveaux dans la théorie des opérations linéaires I.* C. R. Acad. Sci. Paris **239** (1954), 577-579.

[G6] A. Grothendieck, *Résultats nouveaux dans la théorie des opérations linéaires II.* C. R. Acad. Sci. Paris **239** (1954), 607-609.

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Alexander Grothendieck was born in Berlin on 28 March 1928. His father, Sascha Shapiro, an anarchist originally from Russia, took an active part in the revolutionary movements rst in Russia, and then in Germany, during the 1920s, where he met Hanka Grothendieck, Alexanderâ€™s mother.Â While the research topics of the early 1950s were those of functional analysis, the great themes of algebraic geometry, its foundations, such as the redenition of the concept of space itself, occupied the years 1957â€“1970.Â The layman who approaches Grothendieckâ€™s mathematical work has to get past the usual concept of a mathematician as a problem solver and try instead to see mathematics as an art and the mathematician as an artist.