

# **SYMMETRY IN PHYSICS**

## **VOLUME 1:**

### **PRINCIPLES AND SIMPLE APPLICATIONS**

**J. P. ELLIOTT and P. G. DAWBER**

*School of Mathematical and Physical Sciences  
University of Sussex, Brighton*

**OXFORD UNIVERSITY PRESS**  
New York

© J. P. Elliott and P. G. Dawber 1979

All rights reserved. No part of this publication may be reproduced or transmitted, in any form or by any means, without permission.

First published in Great Britain 1979 by  
The Macmillan Press Ltd

First published in paperback 1984  
Reprinted 1986

Published in the U.S.A. by  
Oxford University Press, New York

---

**Library of Congress Cataloging in Publication Data**

---

Elliott, James Philip, 1929-  
Symmetry in physics.

Includes bibliographies and index.

CONTENTS: v. 1. Principles and simple applications.  
-v. 2. Further applications.

1. Symmetry (Physics) I. Dawber, P. G., joint  
author. II. Title.

QC174.17.S9E44 530.1'2 79-12055

ISBN 0-19-520455-7 (v. 1)

ISBN 0-19-520456-5 (v. 2)

---

Printed in Hong Kong

# Contents of Volume 1

<i>Preface</i>	xvii
1 Introduction	1
1.1 The place of symmetry in physics	1
1.2 Examples of the consequences of symmetry	3
1.2.1 One particle in one dimension (classical)	3
1.2.2 One particle in two dimensions (classical)	3
1.2.3 Two particles connected by springs (classical)	4
1.2.4 One particle in three dimensions using quantum mechanics—spherical symmetry and degeneracies	5
1.2.5 One particle in one dimension using quantum mechanics—parity and selection rules	6
1.2.6 The search for symmetry—elementary particle physics	7
1.3 Summary	8
2 Groups and Group Properties	9
2.1 Definition of a group	9
2.2 Examples of groups	11
2.3 Isomorphism	16
2.4 Subgroups	17
2.5 The direct product group	17

2.6	Conjugate elements and classes	18
2.7	Examples of classes	19
2.7.1	The rotation group $\mathcal{R}_3$	19
2.7.2	The finite group of rotations $D_3$	20
2.7.3	The symmetric group $\mathcal{S}_3$	21
2.8	The class structure of product groups	21
2.9	The group rearrangement theorem	22
	<i>Bibliography</i>	22
	<i>Problems</i>	22
3	Linear Algebra and Vector Spaces	24
3.1	Linear vector space	25
3.2	Examples of linear vector spaces	27
3.2.1	Displacements in three dimensions	27
3.2.2	Displacement of a set of $N$ particles in three dimensions	28
3.2.3	Function spaces	28
3.2.4	Function space with finite dimension	29
3.2.5	Wave functions	29
3.3	Linear operators	30
3.4	The multiplication, inverse and transformation of operators	32
3.5	The adjoint of an operator—unitary and Hermitian operators	34
3.6	The eigenvalue problem	35
3.7	Induced transformation of functions	36
3.8	Examples of linear operators	38
3.8.1	Rotation of vectors in the $xy$ -plane	38
3.8.2	Permutations	39
3.8.3	Multiplication by a function in function space	39
3.8.4	Differentiation in function space	40
3.8.5	Induced transformation of functions	40
3.8.6	Further example of induced transformation of functions	41
3.8.7	Transformed operator	41
	<i>Bibliography</i>	42
	<i>Problems</i>	42
4	Group Representations	43
4.1	Definition of a group representation	43
4.2	Matrix representations	44
4.3	Examples of representations	45
4.3.1	The group $D_3$	45
4.3.2	The group $\mathcal{R}_2$	46
4.3.3	Function spaces	47
4.4	The generation of an invariant subspace	48
4.5	Irreducibility	50
4.6	Equivalent representations	52

Contents		vii
4.6.1	Proof of Maschke's theorem	53
4.7	Inequivalent irreducible representations	54
4.8	Orthogonality properties of irreducible representations	54
4.8.1	Proof of Schur's first lemma	58
4.8.2	Proof of Schur's second lemma	60
4.9	Characters of representations	60
4.10	Orthogonality relation for characters of irreducible representations	61
4.11	Use of group characters in reducing a representation	62
4.12	A criterion for irreducibility	63
4.13	How many inequivalent irreducible representations?—the regular representation	64
4.14	The second orthogonality relation for group characters	66
4.15	Construction of the character table	67
4.16	Orthogonality of basis functions for irreducible representations	68
4.17	The direct product of two representations	70
4.18	Reduction of an irreducible representation on restriction to a subgroup	73
4.19	Projection operators	74
4.20	Irreducible sets of operators and the Wigner–Eckart theorem	78
4.21	Representations of direct product groups	81
	<i>Bibliography</i>	83
	<i>Problems</i>	83
5	Symmetry in Quantum Mechanics	85
5.1	Brief review of the framework of quantum mechanics	85
5.2	Definition of symmetry in a quantum system	89
5.3	Degeneracy and the labelling of energies and eigenfunctions	90
5.4	Selection rules and matrix elements of operators	91
5.5	Conservation laws	92
5.6	Examples	93
5.6.1	Symmetry group $C_3$	93
5.6.2	Symmetry group $D_3$	95
5.6.3	Symmetry group $S_2$	96
5.6.4	Symmetry group $\mathcal{R}_2$	96
5.7	Use of group theory in a variational approximation	97
5.8	Symmetry-breaking perturbations	99
5.8.1	Examples	100
5.8.2	Magnitude of the splitting	101
5.9	The indistinguishability of particles	102
5.10	Complex conjugation and time-reversal	103
	<i>Bibliography</i>	104
	<i>Problems</i>	104
6	Molecular Vibrations	106

6.1	The harmonic approximation	107
6.2	Classical solution	108
6.3	Quantum mechanical solution	109
6.4	Effects of symmetry in molecular vibrations	110
6.5	Classification of the normal modes	113
6.5.1	The water molecule	115
6.5.2	The ammonia molecule	116
6.6	Vibrational energy levels and wave functions	117
6.7	Infrared and Raman absorption spectra of molecules	120
6.7.1	Infrared spectra	120
6.7.2	Raman spectra	121
6.8	Displacement patterns and frequencies of the normal modes	122
	<i>Bibliography</i>	124
	<i>Problems</i>	124
7	Continuous Groups and their Representations, Including Details of the Rotation Groups $\mathcal{R}_2$ and $\mathcal{R}_3$	125
7.1	General remarks	126
7.2	Infinitesimal operators	127
7.3	The group $\mathcal{R}_2$	130
7.3.1	Irreducible representations	131
7.3.2	Character	131
7.3.3	Multiplication of representations	132
7.3.4	Examples of basis vectors	132
7.3.5	Infinitesimal operators	133
7.4	The group $\mathcal{R}_3$	134
7.4.1	Infinitesimal operators	135
7.4.2	Irreducible representations	137
7.4.3	Characters	140
7.4.4	Multiplication of representations	141
7.4.5	Examples of basis vectors	143
7.4.6	Irreducible sets of operators and the Wigner–Eckart theorem	146
7.4.7	Equivalent operators	147
7.5	The Casimir operator	148
7.6	Double-valued representations	150
7.7	The complex conjugate representation	153
	<i>Bibliography</i>	153
	<i>Problems</i>	154
8	Angular Momentum and the Group $\mathcal{R}_3$ with Illustrations from Atomic Structure	156
8.1	Rotational invariance and its consequences	156
8.2	Orbital angular momentum of a system of particles	158
8.3	Coupling of angular momenta	159
8.4	Intrinsic spin	161
8.5	The hydrogen atom	166

Contents		ix
8.6	The structure of many-electron atoms	170
8.6.1	The Hamiltonian	170
8.6.2	The Pauli principle and shell filling	171
8.6.3	Atoms with more than one valence electron -- <i>LS</i> coupling	173
8.6.4	Classification of terms	176
8.6.5	Ordering of terms	179
	<i>Bibliography</i>	181
	<i>Problems</i>	181
9	Point Groups with an Application to Crystal Fields	183
9.1	Point-group operations and notation	184
9.2	The stereogram	184
9.3	Enumeration of the point groups	186
9.3.1	Proper groups	186
9.3.2	Improper groups	191
9.4	The class structure of the point groups	192
9.4.1	Proper point groups	193
9.4.2	Improper point groups	193
9.5	The crystallographic point groups	196
9.6	Irreducible representations for the point groups	197
9.7	Double-valued representations of the point groups	199
9.8	Time-reversal and magnetic point groups	201
9.9	Crystal field splitting of atomic energy levels	202
9.9.1	Definition of the physical problem	202
9.9.2	Deduction of the manner of splitting from symmetry considerations	204
9.9.3	Effect of a magnetic field	209
	<i>Bibliography</i>	210
	<i>Problems</i>	211
10	Isospin and the Group $SU_2$	213
10.1	Isospin in nuclei	214
10.1.1	Isospin labelling and degeneracies	215
10.1.2	Splitting of an isospin multiplet	218
10.1.3	Selection rules	221
10.2	Isospin in elementary particles	222
10.2.1	Collisions of $\pi$ -mesons with nucleons	223
10.3	Isospin symmetry and charge-independence	223
	<i>Bibliography</i>	224
	<i>Problems</i>	224
11	The Group $SU_3$ with Applications to Elementary Particles	226
11.1	Compilation of some relevant data	227
11.2	The hypercharge	230
11.3	Baryon number	231
11.4	The group $SU_3$	232
11.5	Subgroups of $SU_3$	233

x

## Contents

11.6	Irreducible representations of $SU_3$	233
11.6.1	Complex conjugate representations	241
11.6.2	Multiplication of representations	242
11.7	Classification of the hadrons into $SU_3$ multiplets	243
11.8	The mass-splitting formula	244
11.9	Electromagnetic effects	247
11.10	Casimir operators	248
	<i>Bibliography</i>	249
	<i>Problems</i>	249
12	Supermultiplets in Nuclei and Elementary Particles—the Groups $SU_4$ and $SU_6$ and Quark Models	251
12.1	Supermultiplets in nuclei	252
12.2	Supermultiplets of elementary particles	255
12.3	The three-quark model	257
12.4	The nine-quark model	260
12.5	Charm	262
	<i>Addendum (mid-1978)</i>	262
	<i>Addendum (late 1983)</i>	263
	<i>Bibliography</i>	264
	<i>Problems</i>	264
Appendix 1	Character Tables for the Irreducible Representations of the Point Groups	265
Appendix 2	Solutions to Problems in Volume 1	275
	<i>Index to Volumes 1 and 2 (adjacent to p. 280)</i>	I



## Contents of Volume 2

	<i>Preface</i>	xvii
13	Electron States in Molecules	281
	13.1 Linear combinations of atomic orbitals (LCAO)	282
	13.2 Examples	284
	13.3 Selection rules for electronic excitations in molecules	287
	<i>Bibliography</i>	288
	<i>Problems</i>	288
14	Symmetry in Crystalline Solids	289
	14.1 Translational symmetry in crystals	289
	14.2 The translation group $\mathcal{T}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$	290
	14.3 The Brillouin zone and some examples	293
	14.4 Electron states in a periodic potential	294
	14.4.1 The nearly-free electron model	295
	14.4.2 Metals and insulators	299
	14.4.3 The tight-binding method	302
	14.5 Lattice vibrations	306
	14.5.1 The one-dimensional monatomic lattice	306
	14.5.2 Three-dimensional crystals with several atoms per unit cell	309
	14.6 Spin waves in ferromagnets	311

14.7	Excitons in insulators (Frenkel excitons)	313
14.8	Selection rules for scattering	314
14.9	Space groups	315
14.9.1	Irreducible representations of space groups	316
14.9.2	Application to electron states	320
14.9.3	Other excitations	323
	<i>Bibliography</i>	323
	<i>Problems</i>	324
15	Space and Time	325
15.1	The Euclidean group $\mathcal{E}_3$	326
15.1.1	Translations	326
15.1.2	The group operators	328
15.1.3	The irreducible representations	328
15.1.4	The group $\mathcal{E}_2$	331
15.1.5	The physical significance of the Euclidean group $\mathcal{E}_3$	331
15.1.6	Scalar products and normalisation of basis vectors	333
15.2	The Lorentz group $\mathcal{L}$	334
15.2.1	The Lorentz transformation	335
15.2.2	The regions of space-time	339
15.2.3	Physical interpretation of the Lorentz transformation	340
15.2.4	Infinitesimal operators	343
15.2.5	The irreducible representations	344
15.3	The Lorentz group with space inversions $\mathcal{L}_s$	347
15.4	Translations and the Poincaré group $\mathcal{P}$	349
15.4.1	Translations in space-time	349
15.4.2	The Poincaré group and its representations	351
15.4.3	Casimir operators	356
15.4.4	Definition of scalar product	359
15.5	The Poincaré group with space inversions $\mathcal{P}_s$	360
15.6	The Poincaré group with time inversion $\mathcal{P}_t$	362
15.7	Physical interpretation of the irreducible representations of the Poincaré group	363
15.7.1	Mass	364
15.7.2	Spin	366
15.7.3	Parity	368
15.7.4	Time-reversal	369
15.7.5	Some consequences of time-reversal symmetry	373
15.8	Single-particle wave functions and the wave equations	375
15.8.1	The group $\mathcal{R}_3$	376
15.8.2	The group $\mathcal{E}_3$	377
15.8.3	The Poincaré group with $s = 0$ —the Klein-Gordon equation	379
15.8.4	The Poincaré group with $s = \frac{1}{2}$ —the Dirac equation	380

Contents		xiii
	15.8.5 Particles with zero mass and spin $ m  = \frac{1}{2}$ —the Weyl equation	387
	15.8.6 Particles with zero mass and spin $ m  = 1$ —the Maxwell equations	389
	<i>Bibliography</i>	390
	<i>Problems</i>	391
16	Particles, Fields and Antiparticles	393
	16.1 Classical mechanics of particles	394
	16.1.1 Lagrange formalism	394
	16.1.2 Hamiltonian formalism	394
	16.1.3 Examples from relativistic mechanics	396
	16.2 Classical mechanics of fields	398
	16.2.1 The transformation of fields	398
	16.2.2 The Lagrange equation for fields	399
	16.2.3 The electromagnetic field	400
	16.3 Quantum fields	401
	16.3.1 Second quantisation	402
	16.3.2 Field operators	404
	16.3.3 The physical role of field operators	405
	16.3.4 Causality and the spin-statistics theorem	408
	16.3.5 Antiparticles	409
	16.3.6 Charge conjugation and the PCT theorem	411
	16.3.7 Field for particles with non-zero spin	413
	<i>Bibliography</i>	423
	<i>Problems</i>	423
17	The Symmetric Group $\mathcal{S}_n$	425
	17.1 Cycles	426
	17.2 The parity of a permutation	427
	17.3 Classes	428
	17.4 The identity and alternating representations—symmetric and antisymmetric functions	430
	17.5 The character table for irreducible representations	431
	17.6 Young diagrams	434
	17.7 The restriction from $\mathcal{S}_n$ to $\mathcal{S}_{n-1}$	434
	17.8 The basis vectors of the irreducible representations	436
	17.9 Examples of basis vectors and representation matrices	438
	17.10 The direct product of two representations	439
	17.11 The outer product of two irreducible representations	441
	17.12 Restriction to a subgroup and the outer product	443
	17.13 The standard matrices of the irreducible representations	445
	17.14 The class operator $\sum_{i < j} T(P_{ij})$	450
	<i>Bibliography</i>	450
	<i>Problems</i>	451
18	The Unitary Group $U_N$	452

xiv

## Contents

18.1	The irreducible representations of $U_N$	453
18.2	Some examples	456
18.3	The chain of subgroups $U_N \rightarrow U_{N-1} \rightarrow U_{N-2} \rightarrow \dots \rightarrow U_2 \rightarrow U_1$	457
18.4	A labelling system for the basis vectors	459
18.5	The direct product of representations of $U_N$	461
18.6	The restriction from $U_N$ to its subgroup $SU_N$	462
18.7	The special cases of $SU_2$ , $SU_3$ and $SU_4$	464
18.8	The infinitesimal operators of $U_N$	466
18.9	The complex conjugate representations of $U_N$ and $SU_N$	467
18.10	The use of the group $U_N$ in classifying many-particle wave functions	469
18.10.1	The use of subgroups of $U_N$	471
18.11	Characters	475
18.12	Group integration and orthogonality	476
18.13	The groups $SU_2$ and $\mathcal{R}_3$	478
18.13.1	The parameters of $SU_2$	478
18.13.2	Infinitesimal operators and irreducible representations of $SU_2$	480
18.13.3	Connection between the groups $\mathcal{R}_3$ and $SU_2$	480
18.13.4	Explicit formula for the parameters of a product of rotations	482
18.13.5	Examples of $SU_2$ basis vectors	482
	<i>Bibliography</i>	483
	<i>Problems</i>	483
19	Two Familiar 'Accidental' Degeneracies—the Oscillator and Coulomb Potentials	485
19.1	The three-dimensional harmonic oscillator for one particle	486
19.2	The three-dimensional harmonic oscillator for many particles	491
19.3	The harmonic oscillator in $n$ dimensions	492
19.4	The symmetry group of the Coulomb potential	492
19.4.1	The groups $\mathcal{R}_4$ and $\mathcal{L}$	494
19.4.2	The classification of states of the Coulomb potential	495
	<i>Bibliography</i>	496
	<i>Problems</i>	497
20	A Miscellany	498
20.1	Non-invariance groups	498
20.2	The Jahn–Teller effect and spontaneously broken symmetries	502
20.2.1	The adiabatic approximation	502
20.2.2	The role of symmetry	503
20.2.3	Spontaneous symmetry breaking	505
20.3	Normal subgroups, semi-direct products and little groups	507

Contents		xv
20.4	The classification of Lie groups	510
20.5	The rotation matrices	519
	<i>Bibliography</i>	522
	<i>Problems</i>	523
Appendix 3	Topics in Representation Theory	524
	A.3.1 Symmetrised products of representations	524
	A.3.2 Use of a subgroup in reducing product representations	527
	A.3.3 Class multiplication	529
Appendix 4	Some Results Pertaining to the Group $\mathcal{R}_3$	531
	A.4.1 An integral over three spherical harmonics	531
	A.4.2 The spherical harmonic addition theorem	532
	A.4.3 Group integration	533
Appendix 5	Techniques in Atomic Structure Calculations	539
	A.5.1 Term energies for $p^2$ and $p^3$ configurations	539
	A.5.2 Recoupling coefficients ( $6j$ - and $9j$ - symbols)	543
	A.5.3 Transition strengths	547
	A.5.4 The crystal field potential	549
	A.5.5 Use of symmetry to deduce ratios of splittings	550
	<i>Problems on appendices 4 and 5</i>	553
Appendix 6	Solutions to Problems in Volume 2	555
	<i>Index to Volumes 1 and 2 (adjacent to p. 558)</i>	I

## Preface to Volume 1

One cannot study any physical system for very long before finding regularities or symmetries which demand explanation and, even though the system may be complex, one expects that the regularities will have a simple explanation. This basic optimism, which pervades not only physics but science in general, is justified in the case of symmetries because there is a theory of symmetry which has application in almost all branches of physics and especially in quantum physics. The object of our book is to describe the theory of symmetry and to study its applications in a wide variety of physical systems.

The book has grown out of several lecture courses which we have given at the University of Sussex during the past ten years. One was a general introductory course on symmetry given to third-year undergraduates, one a postgraduate course on symmetry in solid-state physics and one a postgraduate course on symmetry in atomic, nuclear and elementary-particle physics. As a result, the book may be used by students in any of these categories. We regard chapters 1–5 (inclusive) as a minimum selection for any student wishing to study symmetry, although those students who have taken an undergraduate course on linear algebra will find that much of chapter 3 is familiar and may be read quite rapidly. The remaining chapters 6–12 in volume 1 cover a wide range of applications which is quite sufficient for an undergraduate course. One could even be selective within the first volume by omitting chapters 10–12 on nuclear and elementary particle physics or

xvii

alternatively by omitting chapters 6 and 9 on the point groups. We would expect the second volume to be used for serious study at the postgraduate level and for occasional reference by the more inquisitive undergraduate.

The first chapter of volume 1 introduces the concept of symmetry with some very simple examples and lists the general consequences. We then leave physics aside for three chapters while preparing the mathematical tools to be used later. The most important of these are group theory and linear algebra which are described in chapters 2 and 3. The fourth chapter brings together these two ideas in a study of group representations and it is this aspect of group theory which is most used in the theory of symmetry. We return to physics in chapter 5 with a brief summary of the basic ideas of quantum mechanics and a general description of the effects of symmetry in quantum systems. The remainder of the book is concerned with applications to different physical systems and the study in greater detail of the relevant groups. We cover a broad range of applications from molecular vibrations to elementary particles and in each case we aim to introduce sufficient background description to enable the reader who has no prior knowledge of that particular physical system to appreciate the role being played by symmetry. Each application is reasonably self-contained and the more sophisticated systems are left until the later chapters. The vibration of molecules is the first phenomenon studied in detail, in chapter 6, and here we are able to illustrate the results of symmetry in classical mechanics before going over to the quantised theory. Chapters 7 and 8 describe the symmetry with respect to rotations with applications to the structure of atoms. It is here that we meet for the first time a continuous group, with an infinite number of elements, or symmetry operations, and the general properties of such groups are described. Chapter 9 describes in some detail the 'point groups', which contain only a finite number of rotations, and uses them to study the influence of a crystal field on atomic states. In chapters 10, 11 and 12 we move on to the more abstract symmetries encountered in nuclear and elementary particle physics but make use of the same general theory that was used for the more concrete applications in earlier chapters. We introduce the groups of unitary transformations in two, three, four and six dimensions and use them to describe the observed symmetry between neutrons and protons and the regularities amongst some of the recently discovered short-lived elementary particles. The ideas of 'strangeness' and 'quarks' are explained.

Volume 2 begins with a further application of the use of 'point groups'—to the motion of electrons in a molecule—and then, in chapter 14, moves away from symmetries with a fixed point to study discrete translations and their applications to crystal structure. The theory of relativity is of profound importance in the philosophy of physics and, when speeds become comparable with that of light, it has practical importance. For all the systems discussed in volume 1 we are able to ignore relativity because the speeds of the particles involved are sufficiently small. Chapter 15 describes the symmetry in four-dimensional space-time which is the origin of relativity theory and discusses its consequences, especially in relation to the classification of elementary

## Preface

xix

particles. The concepts of momentum, energy, mass and spin are interpreted in terms of symmetry using the Lorentz and Poincaré groups and a natural place is found in the theory for particles, like the photon, with zero mass. Chapter 16 is concerned with fields, in contrast to the earlier chapters which dealt with particles or systems of particles. We first describe classical fields, such as the electromagnetic field, using four-dimensional space-time. This is followed by a brief account of the theory of relativistic quantum fields which provides a framework for the creation and annihilation of particles and the existence of antiparticles. Chapters 17 and 18 contain details of two general groups, the 'symmetric' group of all permutations of  $n$  objects and the 'unitary' group in  $N$  dimensions, and an intimate relation between these two groups is discussed. Particular cases of these two groups have been met earlier. Chapter 19 describes some unexpected symmetries in two familiar potentials, the Coulomb and the harmonic oscillator potentials, and a number of small, unconnected, but interesting topics are collected into the last chapter.

The text includes worked examples and a selection of problems with solutions. A bibliography of references for further reading is given at the end of each chapter for those who wish either to follow the physical applications into more detail or to study some of the mathematical questions to a greater depth.

To aid the reader we have followed the standard convention of using italic type for algebraic symbols such as  $x$ ,  $y$  and  $z$ , whereas operators are distinguished by the use of roman type. An operator or matrix will be written  $T$  but its matrix elements  $T_{ij}$ , which are numbers, will be in italic type. In addition, bold face type will be used for vectors and in chapters 15 and 16 of volume 2 we meet four-vectors  $\hat{e}$  which are printed with a circumflex.

Brighton, Sussex, 1979

J. P. E.  
G. G. D.



# 1

## Introduction

### 1.1 The place of symmetry in physics

According to the *Concise Oxford Dictionary*, symmetry is defined as '(Beauty resulting from) right proportion between the parts of the body of any whole, balance, congruity, harmony, keeping'. Although there is much complex detail in physics there is also much beauty and simplicity and it is the symmetry in physical laws and physical systems which is largely responsible for this. Consequently, symmetry plays an important role in physics and one which is increasing in importance with modern developments. It is the purpose of this book to explain in general terms why the existence of symmetry leads to a variety of physical simplicities in both classical and quantum mechanics. To illustrate the general results we shall refer to simple properties of molecules, crystals, atoms, nuclei and elementary particles. Although these physical systems are so obviously different from one another, nevertheless the same theory of symmetry may be applied to them all. The study of symmetry, therefore, helps to unify physics by emphasising the similarity between different fields.

It is true that symmetry plays a part in both classical and quantum physics, but it is in the latter that most interest lies. There are several reasons for this. The first is that there is a much greater scope for symmetry to exist in the microscopic domain since, for example, one electron is identical with any other

electron and one atom of carbon (say) is identical with any other. The second reason is that at the microscopic level one must use quantum mechanics which is inherently more complicated than classical mechanics and so provides more scope for simplification through symmetry arguments. For example, a particle is described by a wave *function* rather than a single position. One further reason is that the structure of atomic and subatomic systems is now one of the exciting frontiers of science and the ideas of symmetry are helping to create order out of apparent chaos.

Throughout physics one uses mathematics as the tool with which to investigate the consequences of some assumed theory or model. For example, in the motion of a particle of mass  $M$  in one dimension  $x$  under some force  $f(x)$  the physical law (Newtonian theory) tells us that  $f(x) = M(d^2x/dt^2)$ . To find the position  $x(t)$ , as a function of time, given  $f(x)$ , we must solve this differential equation, putting in the initial values of  $x$  and  $dx/dt$ . Thus, in Newtonian mechanics, the differential and integral calculus is the appropriate tool. In studying the symmetry of physical systems we are asking about their behaviour under transformations. For example, if a particle moves in one dimension under the influence of a potential  $V(x)$ , that potential may have reflection symmetry in the origin, i.e.  $V(-x) = V(x)$ . In this case the potential is said to be invariant (unchanged) under the transformation which replaces  $x$  by  $-x$ . In another example, that of a particle moving in three dimensions, the potential may have spherical symmetry, which means that, in spherical polar coordinates, the potential is independent of angle and may be written  $V(r)$ . In this case the potential is invariant under any of the transformations which rotate through any angle about any axis through the origin—an infinite number of transformations!

To investigate the physical consequences of the symmetry of a system we must, therefore, learn something about transformations and in particular about the set (collection) of transformations which leave some function, like the potential, invariant. The theory of such sets of transformations is called 'group theory' by mathematicians and this is the appropriate tool for the physicist to use in studying symmetry.

It is fascinating to draw an analogy between the use of calculus in classical mechanics and the use of group theory in quantum mechanics. Historically the discovery of Newton's laws and the invention of the calculus occurred at about the same time in the seventeenth century. Although the ideas of group theory were introduced into mathematics as early as 1810 it was not until the 1920s that the theory of group representations, which is crucial to the study of symmetry, was developed. This was the very time when physicists were formulating the quantum theory. In fact the significance of symmetry in quantum mechanics was realised very early in the classic works of E. Wigner, in 1931, H. Weyl, in 1928, and Van-der-Waerden, in 1932.

There have always been those who have argued that it is unnecessary to use group theory in quantum mechanics. In a sense this is true, since group theory itself is built from elementary algebraic steps. However, the investment of

effort in learning to use the sophisticated tool which is group theory is soon rewarded by handsome dividends of simplification and unification in the study of complex quantum mechanical systems. After all, one could argue that the calculus is not necessary in classical mechanics. For example, geometrical arguments could be used to show that the inverse square law of gravitational attraction leads to elliptical orbits. In fact, Newton originally used such a method but in modern times we understand this result through the solution of a differential equation. Looking ahead, it is exciting to speculate that further major advances in mathematics and physics may go hand in hand in the future.

## 1.2 Examples of the consequences of symmetry

To whet the appetite we now list a number of physical systems which possess symmetry and we point out some features of their behaviour which are direct consequences of the symmetry. Simpler examples are given first. Although in some cases we are able to relate the behaviour to the symmetry without developing new methods this is, of course, not always possible. It is the purpose of this book to describe generally the consequences of symmetry and it will not be until much later in the book that we shall be in a position to understand and to predict the behaviour of systems with intricate symmetries.

### 1.2.1 One particle in one dimension (classical)

A particle of mass  $M$ , moving in one dimension under the influence of a potential  $V(x)$ , will have its motion governed by the equation

$$M\ddot{x} = -dV/dx \quad (1.1)$$

Suppose now that  $V(x)$  is a constant, independent of  $x$ ; in other words that it is invariant under translation. Then clearly  $M\ddot{x} = 0$  and, integrating, gives  $M\dot{x} = C$ , showing the conservation (constancy) of linear momentum  $M\dot{x}$ .

### 1.2.2 One particle in two dimensions (classical)

In two dimensions the motion of the particle is governed by the two equations

$$M\ddot{x} = -\partial V/\partial x \quad \text{and} \quad M\ddot{y} = -\partial V/\partial y \quad (1.2)$$

Suppose now that  $V(x, y)$  is invariant with respect to rotation about the origin; in other words that  $V(x, y)$  is independent of the polar angle  $\theta$  if expressed in terms of the polar coordinates  $r, \theta$  rather than the cartesian  $x$  and  $y$ . In this case  $\partial V/\partial \theta = 0$ . However,

$$\frac{\partial V}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial V}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial V}{\partial y} = -y \frac{\partial V}{\partial x} + x \frac{\partial V}{\partial y}$$

4

Introduction

1.2.3

and using equation (1.2)

$$\frac{\partial V}{\partial \theta} = M(y\ddot{x} - x\ddot{y}) = M \frac{d}{dt}(y\dot{x} - x\dot{y})$$

so that the invariance  $\partial V/\partial \theta = 0$  implies the constancy of the quantity  $M(y\dot{x} - x\dot{y})$  which is the moment of momentum (or angular momentum) about an axis through the origin and perpendicular to the plane.

If the particle were free to move in three dimensions in a potential which was invariant with respect to rotations about *any* axis then this argument shows that any component of the angular momentum is constant. In other words, for a spherically symmetric potential, both the magnitude and the direction of the angular momentum are conserved.

### 1.2.3 Two particles connected by springs (classical)

Two particles of equal mass  $M$  are connected to each other and to fixed supports by equal collinear springs with spring constant  $\lambda$ . Let the natural length of the springs be  $a$  and the supports a distance  $3a$  apart. Denote the displacements of the two particles from their equilibrium positions by  $x_1$  and  $x_2$ . Although the general displacement, illustrated in figure 1.1, has no

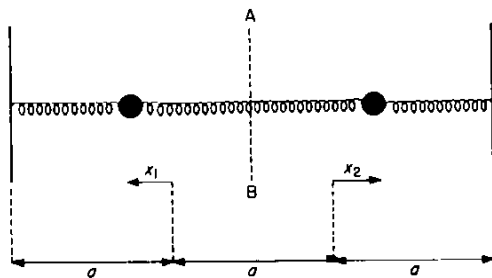


Figure 1.1

symmetry it is intuitively clear that, in some sense, the system has reflection symmetry about the centre. In fact, both the kinetic and potential energies

$$T = \frac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) \quad \text{and} \quad V = \frac{1}{2}\lambda\{x_1^2 + x_2^2 + (x_1 + x_2)^2\}$$

are invariant with respect to the interchange of  $x_1$  and  $x_2$ , which is the transformation of coordinates  $x_1$  and  $x_2$  produced by a reflection in the line AB.

The consequences of symmetry are not very dramatic in this case, but the generalisation to the vibration of atoms about their equilibrium positions in a molecule is of considerable importance. It is therefore worth while to solve

This two-volume set can be naturally divided into two semester courses, and contains a full modern graduate course in quantum physics. The idea is to teach graduate students how to practically use quantum physics and theory, presenting the fundamental knowledge, and gradually moving on to applications, including atomic, nuclear and solid state physics, as well as modern subfields, such as quantum chaos and quantum entanglement. He graduated from Moscow University and worked for many years at the Budker Institute of Nuclear Physics in Novosibirsk where he got his Candidate of Science and highest Doctor of Science degrees (equivalent to a PhD). Fluid dynamics - Zienkiewicz O.C., Taylor R.L. The Properties of Gases and Liquids, Fifth Edition - Poling, Prausnitz, O'Connell Fun : Black Holes Wormholes and Time Machines - Jim Al-Khalili Great Physicists - From Galileo to Hawking - W. Cropper Mad About Modern Physics - F. Potter, C. Jargodzki Mathematical Principles of Natural Philosophy I The Motion of Bodies - I. Newton Perfect Symmetry - The Search for. Numerical Quantum Dynamics - W. Schweizer Path integrals and their applications in quantum, statistical, and solid state physics - Papadopoulos, J. T. Devreese Path Integrals in Physics Volume 1 Stochastic Process Physics from Symmetry. Undergraduate Lecture Notes in Physics. Undergraduate Lecture Notes in Physics (ULNP) publishes authoritative texts covering topics throughout pure and applied physics. Each title in the series is suitable as a basis for undergraduate instruction, typically containing practice problems, worked examples, chapter summaries, and suggestions for further reading. In the course of studying physics I became, like any student of physics, familiar with many fundamental equations and their solutions, but I wasn't really able to see their connection. I was thrilled when I understood that most of them have a common origin: Symmetry.