Linear Quantum Feedback Networks

J.E. Gough R. Gohm M. Yanagisawa

(Dated: June 9, 2009)

The mathematical theory of quantum feedback networks has recently been developed [5] for general open quantum dynamical systems interacting with bosonic input fields. In this article we show, for the special case of linear dynamical Markovian systems with instantaneous feedback connections, that the transfer functions can be deduced and agree with the algebraic rules obtained in the nonlinear case. Using these rules, we derive the transfer functions for linear quantum systems in series, in cascade, and in feedback arrangements mediated by beam splitter devices.

PACS numbers:

I. INTRODUCTION

The aim of this paper is to deduce the algebraic rules for determining the dynamical characteristics of a prescribed network consisting of specified quantum oscillator systems connected by input-output fields [1], [2]. Physical models include cavity systems or local quantum oscillators with a quantum optical field. The resulting dynamics is linear, and the analysis is carried out using transfer function techniques [3], [4]. The rules have been recently deduced in [5] in the general setting for nonlinear quantum dynamical systems by first constructing a network Hamiltonian and transferring to the interaction picture with respect to the free flow of the fields around the network channels. However it is of interest to restrict to linear systems for two main reasons. Firstly, the derivation here for linear systems proceeds by an alternative method to the general nonlinear case, and we are able to confirm the restriction of the nonlinear formula to linear systems yields the same result. Secondly, linear systems are the most tractable and, therefore, most widely studied models in classical control theory and so it is natural to develop these further. There has been recent interest in the development of coherent, or fully quantum control for linear systems [6]-[10] and this paper contributes by establishing the algebraic rules for building networks of such devices.

II. LINEAR QUANTUM MARKOV MODELS

The dynamical evolution of a quantum system is determined by a family of unitaries $\{V(t,s) : t \geq s\}$ satisfying the propagation law $V(t_3,t_2)V(t_2,t_1) = V(t_3,t_1)$ where $t_3 \geq t_2 \geq t_1$. The evolution of a state from time s to a later time t being then given by $\psi(t) = V(t,s)\psi(s)$. In a Markov model we factor the underlying Hilbert space as $\mathfrak{h} \otimes \mathcal{E}$ representing the system and its environment respectively and the unitary $V(t,s)$ couples the system specifically with the degrees of freedom of the environment acting between times s and t. For a bosonic environment, we introduce input processes $b_i(t)$ for $i = 1, \cdots, n$ with the canonical commutation relations, [1],

$$\left[b_i(t), b_j^\dagger(s) \right] = \delta_{ij}\,\delta(t-s). \tag{1}$$

It is convenient to assemble these into the following column vectors of length n

$$\mathbf{b}^{\text{in}}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}. \tag{2}$$

For a Markov evolution $V(t,s)$ can be described equivalently by the chronological-ordered and Wick-ordered expressions

$$\vec{T} \exp -i \int_s^t \Upsilon(\tau)\, d\tau \equiv\, : \exp -i \int_s^t \Upsilon_{\text{Wick}}(\tau)\, d\tau :$$

where the stochastic Hamiltonian is (with $E_{ij}^\dagger = E_{ji}$ and $K^\dagger = K$)

$$\begin{aligned} \Upsilon(t) &= \sum_{i,j=1}^n E_{ij} \otimes b_i^\dagger(t)\, b_j(t) \\ &\quad + \sum_{i=1}^n F_i \otimes b_i^\dagger(t) + \sum_{j=1}^n F_j^\dagger \otimes b_j(t) + K \otimes 1, \end{aligned}$$

and the Wick-ordered generator is given by [13]

$$\begin{aligned} -i\Upsilon_{\text{Wick}}(t) &= \sum_{i,j=1}^n (S_{ij} - \delta_{ij}) \otimes b_i^\dagger(t)\, b_j(t) \\ &\quad + \sum_{i=1}^n L_i \otimes b_i^\dagger(t) - \sum_{i,j=1}^n L_i^\dagger S_{ij} \otimes b_j(t) \\ &\quad - \left(\frac{1}{2}\sum_{i=1}^n L_i^\dagger L_i + iH\right) \otimes 1. \end{aligned}$$

The Wick-ordered coefficients are given by the Stratonovich-Ito conversion formulae, see appendix,

$$\begin{aligned} S &= \frac{1 - \frac{i}{2}E}{1 + \frac{i}{2}E}, \quad L = -i\frac{1}{1 + \frac{i}{2}E}F, \\ H &= K + \frac{1}{2}\text{Im} F \frac{1}{1 + \frac{i}{2}E} F^\dagger. \end{aligned} \tag{3}$$

Note that H is selfadjoint, and that S is a unitary matrix whose entries are operators on $\mathfrak{h}$: $\sum_{k=1}^n S_{ik} S_{jk}^\dagger = \delta_{ij} =$

$\sum_{k=1}^{n} S_{ki}^{\dagger} S_{kj}$. In fact, we may write $S = e^{-iJ}$ with $J = 2\arctan\frac{E}{2}$. In the following, we shall omit the tensor product symbol and write simply X for any operator of the form $X \otimes 1$ acting trivially on the environment space.

In differential form we have (summation over repeated indices)

$$\begin{aligned}\frac{d}{dt}V(t,s) = &-i : \Upsilon_{\text{Wick}}(t)\, V(t,s) : \\ \equiv b_i^{\dagger}(t)\,(S_{ij}-\delta_{ij})V(t,s)\,b_j(t) &+ b_i^{\dagger}(t)\, L_i V(t,s) \\ -L_i^{\dagger} S_{ij} V(t,s)\, b_j(t) &- \left(\frac{1}{2}L_i^{\dagger}L_i - iH\right) V(t,s).\end{aligned}$$

Note that all the creators appear on the left and all annihilators on the right. This equation can be interpreted as a quantum stochastic differential equation [1], [11], [12].

We sketch the system plus field as a two port device having an input and an output port.

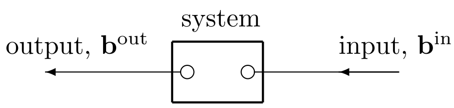

Figure 1: input-output component

The output fields are defined by $b_i^{\text{out}}(t) = V(t,0)^{\dagger}\, b_i(t)\, V(t,0)$ and we have the input-output relation, see appendix for the derivation in the Hudson-Parthasarathy calculus [11],

$$b_i^{\text{out}}(t) = \sum_{j=1}^{n} S_{ij}(t)\, b_j(t) + L_i(t),$$

where $S_{ij}(t) = V(t,0)^{\dagger} S_{ij} V(t,0)$ and $L_i(t) = V(t,0)^{\dagger} L_i V(t,0)$. More compactly, $\mathbf{b}^{\text{out}}(t) = S(t)\,\mathbf{b}^{\text{in}}(t) + L(t)$.

Let X be a fixed operator of the system and set $X(t,t_0) = V(t,t_0)^{\dagger} X V(t,t_0)$, then we obtain the Heisenberg-Langevin equation (summation convention)

$$\frac{d}{dt}X(t,t_0) = V(t,t_0)^{\dagger}\,\frac{1}{i}[X,\Upsilon(t)]V(t,t_0)$$

$$\begin{aligned} = \; & b_i^{\dagger}(t)\, V(t,t_0)^{\dagger}\left(S_{ki}^{\dagger} X S_{kj} - \delta_{ij}X\right) V(t,t_0)\, b_j(t) \\ & + b_i^{\dagger}(t)\, V(t,t_0)^{\dagger} S_{ki}^{\dagger}\,[X, L_k]\, V(t,t_0) \\ & + V(t,t_0)^{\dagger}\,[L_i^{\dagger}, X] S_{ij} V(t,t_0)\, b_j(t) \\ & + V(t,t_0)^{\dagger}\,\{\mathcal{L}(X) - i\,[X,H]\}\, V(t,t_0),\end{aligned}$$

where $\mathcal{L}(X) = \frac{1}{2}L_k^{\dagger}[X, L_k] + \frac{1}{2}[L_k^{\dagger}, X]L_k$. (Again, see appendix for the derivation in the Hudson-Parthasarathy calculus.) Note that the final term does not involve the input noises, and that the expression in braces is a Lindbladian. In the special case where $S = 1$, this equation reduces to the class of Heisenberg-Langevin equations introduced by Gardiner [1].

A. Linear Models

We consider a quantum mechanical system consisting of a family of harmonic oscillators $\{a_j : j = 1, \cdots, m\}$ with canonical commutation relations $[a_j, a_k] = 0 = \left[a_j^{\dagger}, a_k^{\dagger}\right]$ and $\left[a_j, a_k^{\dagger}\right] = \delta_{jk}$. We collect into column vectors:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}. \tag{4}$$

Our interest is in the general linear open dynamical system and this corresponds to the following situation:

1) The S_{jk} are scalars.
2) The $L_j's$ are linear, i.e., there exist constants c_{jk} such that $L_j \equiv \sum_k c_{jk} a_k$.
3) H is quadratic, i.e., there exist constants ω_{jk} such that $H = \sum_{jk} a_j^{\dagger} \omega_{jk} a_k$.

The complex damping is $\frac{1}{2}L^{\dagger}L + iH = -\mathbf{a}^{\dagger} A \mathbf{a}$ where $A = -\frac{1}{2}C^{\dagger}C - i\Omega$ with $C = (c_{jk})$ and $\Omega = (\omega_{jk})$. Note that $\Omega = \Omega^{\dagger}$ because H is selfadjoint, hence $-\frac{1}{2}C^{\dagger}C \leq 0$ is the real part of A.

Lemma 1: The spectrum *spec*(A) of A is contained in the closed left half plane. If C is invertible then the spectrum of A is contained in the open left half plane (we call this the stable case).

Proof: By the Bendixson-Hirsch theorem (see Problem 214 in [14])

$$spec(A) \subset \overline{W(Re(A))} + i\,\overline{W(Im(A))},$$

where $W(\cdot)$ denotes the numerical range of an operator. In our case $Re(A)$ is non-positive (strictly negative if C is invertible) and the result follows. ■

The Heisenberg-Langevin equations for $\mathbf{a}(t) = V(t,0)\,\mathbf{a}V(t,0)$ and input-output relations then simplify down to

$$\dot{\mathbf{a}}(t) = A\mathbf{a}(t) - C^{\dagger}S\mathbf{b}(t), \tag{5}$$

$$\mathbf{b}^{\text{out}}(t) = S\mathbf{b}(t) + C\mathbf{a}(t). \tag{6}$$

These linear equations are amenable to Laplace transform techniques [3],[4]. We define for Re$s > 0$

$$\hat{C}(s) = \int_0^{\infty} e^{-st} C(t)\, dt, \tag{7}$$

where C is now any of our stochastic processes. Note that $\widehat{\dot{\mathbf{a}}}(s) = s\hat{\mathbf{a}}(s) - \mathbf{a}$. We find that

$$\begin{aligned}\hat{\mathbf{a}}(s) &= -(sI_m - A)^{-1} C^{\dagger} S \hat{\mathbf{b}}^{\text{in}}(s) + (sI_m - A)^{-1}\mathbf{a}, \\ \hat{\mathbf{b}}^{\text{out}}(s) &= S\hat{\mathbf{b}}^{\text{in}}(s) + C\hat{\mathbf{a}}(s).\end{aligned}$$

The operator $\hat{\mathbf{a}}(s)$ can be eliminated entirely to give

$$\hat{\mathbf{b}}^{\text{out}}(s) = \Xi(s)\,\hat{\mathbf{b}}^{\text{in}}(s) + \xi(s)\,\mathbf{a} \tag{8}$$

where the *transfer matrix function* is

$$\Xi(s) = S - C\left(sI_m - A\right)^{-1} C^\dagger S \tag{9}$$

and $\xi(s) = C\left(sI_m - A\right)^{-1}$.

As an example, consider a single mode cavity coupling to the input field via $L = \sqrt{\gamma}a$, and with Hamiltonian $H = \omega a^\dagger a$. This implies $A = -(\frac{\gamma}{2} + i\omega)$ and $C = \sqrt{\gamma}$. If the output picks up an additional phase $S = e^{i\phi}$, the corresponding transfer function is then computed to be

$$\Xi_{cavity}(s) = e^{i\phi}\,\frac{s + i\omega - \frac{\gamma}{2}}{s + i\omega + \frac{\gamma}{2}}. \tag{10}$$

B. The Transfer Matrix Function

The models we consider are therefore determined completely by the matrices (S, C, Ω) with $S \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{n\times m}$ and $\Omega \in \mathbb{C}^{m\times m}$. We shall use the convention $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = D + C\left(s - A\right)^{-1} B$ for matrices $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{n\times n}$, and write the transfer matrix function as

$$\Xi(s) = \left[\begin{array}{c|c} A & -C^\dagger S \\ \hline C & S \end{array}\right](s), \tag{11}$$

where $A = -\frac{1}{2}C^\dagger C - i\Omega$. We note the decomposition

$$\Xi = \left[I_n - C\left(sI_m - A\right)^{-1} C^\dagger\right] S \equiv \left[\begin{array}{c|c} A & -C^\dagger \\ \hline C & I_n \end{array}\right] S.$$

In the simplest case of a single cavity mode we have

$$\Xi_{\text{cavity}}(s) = \left[\begin{array}{c|c} -\frac{\gamma}{2} - i\omega & -\sqrt{\gamma}e^{i\phi} \\ \hline \sqrt{\gamma} & e^{i\phi} \end{array}\right](s).$$

Lemma 2: For each $\omega \in \mathbb{R}$, the transfer function $\Xi(i\omega) \equiv \Xi(0^+ + i\omega)$ is unitary whenever it exists.

Proof: The decomposition follows immediately from (11). We have then for instance $\Xi(0^+ + i\omega)\,\Xi(0^+ + i\omega)^\dagger$ equal to

$$\left[I - C\frac{1}{\frac{1}{2}C^\dagger C + i\Omega'}C^\dagger\right]\left[I - C\frac{1}{\frac{1}{2}C^\dagger C - i\Omega'}C^\dagger\right]$$
$$= I - C\frac{1}{\frac{1}{2}C^\dagger C + i\Omega'}X\frac{1}{\frac{1}{2}C^\dagger C - i\Omega'}C^\dagger,$$

where $\Omega' = \Omega + \omega$ and

$$X = \frac{1}{2}C^\dagger C + i\Omega' + \frac{1}{2}C^\dagger C - i\Omega' - C^\dagger C \equiv 0.$$

The relation $\Xi(0^+ + i\omega)\,\Xi(0^+ + i\omega)^\dagger = I$ is similarly established. ■

Lemma 3: If A is a function of $C^\dagger C$ then

$$\Xi(s) = \frac{s + \tilde{A}^\dagger}{s - \tilde{A}}S,$$

where $\tilde{A}$ is a function of $CC^\dagger$ and Ξ may be analytically continued into the whole complex plane.

Proof: Here we must have $A = -\frac{1}{2}C^\dagger C - i\varepsilon\left(C^\dagger C\right)$ where ε is a real-valued function. We set $\tilde{A} = -\frac{1}{2}CC^\dagger - i\varepsilon\left(CC^\dagger\right)$. From the identity $Cf\left(C^\dagger C\right)C^\dagger = f\left(CC^\dagger\right)CC^\dagger$ for suitable analytic functions f, we have $\Xi(s) \equiv \left[I - \frac{1}{s-\tilde{A}}CC^\dagger\right]S$ and so

$$\left(s - \tilde{A}\right)\Xi(s) \;=\; \left(s - \tilde{A} - CC^\dagger\right)S \equiv \left(s + \tilde{A}^\dagger\right)S.$$

■

The hermitean matrices $C^\dagger C$ and $CC^\dagger$ will have the same set of eigenvalues: to see this, suppose that ϕ is a non-zero unit eigenvector of $CC^\dagger$ with eigenvalue γ, then $\psi = \gamma^{-1/2}C^\dagger\phi$ is a unit eigenvector of $C^\dagger C$ with the same eigenvalue, conversely, every eigenvector ψ of $C^\dagger C$ with non-zero eigenvalue γ gives rise to a nonzero eigenvector $\phi = \gamma^{-1/2}C\psi$ of $CC^\dagger$.

Let $CC^\dagger$ have the spectral form $\sum_k \gamma_k E_k$ with real eigenvalues γ_k and corresponding eigen-projectors E_k, giving a resolution of identity $\sum_k E_k = 1$, then we have

$$\Xi(s) = \sum_k \frac{s - \frac{1}{2}\gamma_k + i\varepsilon_k}{s + \frac{1}{2}\gamma_k + i\varepsilon_k}E_k S,$$

where $\varepsilon_k = \varepsilon(\gamma_k)$. In particular, the rational fraction is of modulus unity for imaginary $s\,(= i\omega)$ and we may write

$$\Xi\left(0^+ + i\omega\right) = \sum_k e^{i\phi_k(\omega)}E_k S$$

where $\phi_k(\omega) = \arg\frac{i(\omega+\varepsilon_k)-\gamma_k/2}{i(\omega+\varepsilon_k)+\gamma_k/2}$. Note that $\Xi(0^+ + i\omega)$ is clearly unitary and the limit $\omega \to 0$ is well-defined. This limit will equal $-S$ in the special case that A is selfadjoint (i.e., $\varepsilon \equiv 0$). Ξ may be analytically continued into the negative-real part of the complex plane. The poles of Ξ then form the resolvent set of $\tilde{A}$, and the zeroes are obtained by reflection about the imaginary axis.

C. Gauge Invariance

We have not considered the most general form of linear model here. Let T be a unitary $m\times m$ matrix and consider the transformation $\mathbf{a}' = T\mathbf{a}$ defining new canonical operators. We find that $L = C'\mathbf{a}'$, $H = \mathbf{a}'^\dagger\Omega'\mathbf{a}'$ where

$C' = CT^{\dagger}$ and $\Omega' = T\Omega T^{\dagger}$. The equations are covariant under these transformations and we have the equivalence

$$\left[\begin{array}{c|c} A & -C^{\dagger}S \\ \hline C & S \end{array}\right] = \left[\begin{array}{c|c} TAT^{\dagger} & -TC^{\dagger}S \\ \hline CT^{\dagger} & S \end{array}\right].$$

If we have however, $L_i = \sum_j c_{ij}a_j + \sum_j d_{ij}a_j^{\dagger}$ and/or $H = \sum_{ij}(\omega_{ij}a_i^{\dagger}a_j + \varepsilon_{ij}a_ia_j + \varepsilon_{ij}a_i^{\dagger}a_j^{\dagger})$ we then generate a linear (now in terms of both the creator and annihilator operators), but non gauge invariant dynamics. This is of importance in discussing issues such as squeezing, but we will not pursue this in this paper.

III. INTRODUCING CONNECTIONS

The situation depicted in the figure below is one where (some of) the output channels are fed back into the system as an input. Prior to the connection between output port(s) s_{i} and input port(s) r_{i} being made, we may model the component as having the total input $\mathbf{b}^{\text{in}} = \begin{pmatrix} \mathbf{b}_{\mathsf{i}}^{\text{in}} \\ \mathbf{b}_{\mathsf{e}}^{\text{in}} \end{pmatrix}$ and total output $\mathbf{b}^{\text{out}} = \begin{pmatrix} \mathbf{b}_{\mathsf{i}}^{\text{out}} \\ \mathbf{b}_{\mathsf{e}}^{\text{out}} \end{pmatrix}$ where the $\mathbf{b}_{\mathsf{j}}^{\text{in}}$ and $\mathbf{b}_{\mathsf{j}}^{\text{out}}$ may be multi-dimensional noises (we in fact only require the multiplicities to agree for $\mathsf{j} = \mathsf{i}, \mathsf{e}$ respectively).

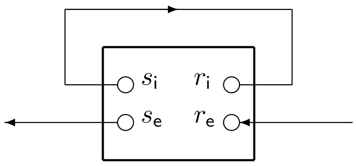

figure 2: A quantum system with feedback

The transfer matrix function takes the general form

$$\Xi \equiv \left[\begin{array}{c|cc} A & -\sum_{\mathsf{j}} C_{\mathsf{j}}^{\dagger}S_{\mathsf{ji}} & -\sum_{\mathsf{j}} C_{\mathsf{j}}^{\dagger}S_{\mathsf{je}} \\ \hline C_{\mathsf{i}} & S_{\mathsf{ii}} & S_{\mathsf{ie}} \\ C_{\mathsf{e}} & S_{\mathsf{ei}} & S_{\mathsf{ee}} \end{array}\right].$$

When we make the connection, we impose the various constraints $b_{r_{\mathsf{i}}(k)}^{\text{in}}(t) = b_{s_{\mathsf{i}}(j)}^{\text{out}}(t-\tau)$ where output field labelled $s_{\mathsf{i}}(j)$ is to be connected to the input field $r_{\mathsf{i}}(k)$ where $\tau > 0$ is the time delay. We assume the idealized situation of instantaneous feedback $\tau \to 0^+$. To avoid having to match up the labels of the internal channels, it is more convenient to introduce a fixed labelling and write

$$\mathbf{b}_{\mathsf{i}}^{\text{out}}(t^-) = \eta \mathbf{b}_{\mathsf{i}}^{\text{in}}(t)$$

where η is the adjacency matrix:

$$\eta_{sr} = \begin{cases} 1, & \text{if } (s,r) \text{ is an internal channel,} \\ 0, & \text{otherwise.} \end{cases}$$

The model with connections is then a reduction of the original and the remaining external fields are the inputs $\mathbf{b}_{\mathsf{e}}^{\text{in}}$ and the outputs $\mathbf{b}_{\mathsf{e}}^{\text{out}}$.

Theorem 1: Let $(\eta - S_{\mathsf{ii}})$ be invertible. The feedback system described above has input-output relation $\hat{\mathbf{b}}_{\mathsf{e}}^{\text{out}} = \Xi_{\text{red}}\hat{\mathbf{b}}_{\mathsf{e}}^{\text{in}} + \xi_{\text{red}}\mathbf{a}$ and the reduced transfer matrix function

$$\Xi_{\text{red}} \equiv \left[\begin{array}{c|c} A_{\text{red}} & -C_{\text{red}}^{\dagger}S_{\text{red}} \\ \hline C_{\text{red}} & S_{\text{red}} \end{array}\right], \quad \xi_{\text{red}} \equiv C_{\text{red}}\frac{1}{s - A_{\text{red}}},$$

where

$$\begin{aligned} S_{\text{red}} &= S_{\mathsf{ee}} + S_{\mathsf{ei}}(\eta - S_{\mathsf{ii}})^{-1}S_{\mathsf{ie}}, \\ C_{\text{red}} &= S_{\mathsf{ei}}(\eta - S_{\mathsf{ii}})^{-1}C_{\mathsf{i}} + C_{\mathsf{e}}, \\ A_{\text{red}} &= A - \sum_{\mathsf{j}=\mathsf{i},\mathsf{e}} C_{\mathsf{j}}^{\dagger}S_{\mathsf{ji}}(\eta - S_{\mathsf{ii}})^{-1}C_{\mathsf{i}}. \end{aligned} \tag{12}$$

Proof: The dynamical equations can be written as

$$\begin{aligned} \dot{\mathbf{a}}(t) &= A\mathbf{a}(t) - \sum_{\mathsf{j},\mathsf{k}} C_{\mathsf{j}}^{\dagger}S_{\mathsf{jk}}\mathbf{b}_{\mathsf{k}}^{\text{in}}(t), \\ \mathbf{b}_{\mathsf{j}}^{\text{out}}(t) &= \sum_{k=\mathsf{i},\mathsf{e}} S_{\mathsf{jk}}\mathbf{b}_{\mathsf{k}}^{\text{in}}(t) + C_{\mathsf{j}}\mathbf{a}(t). \end{aligned}$$

Now the constraint $\eta\mathbf{b}_{\mathsf{i}}^{\text{in}} = \mathbf{b}_{\mathsf{i}}^{\text{out}}$ implies that

$$\mathbf{b}_{\mathsf{i}}^{\text{in}}(t) = (\eta - S_{\mathsf{ii}})^{-1}(S_{\mathsf{ie}}\mathbf{b}_{\mathsf{e}}^{\text{in}}(t) + C_{\mathsf{i}}\mathbf{a}(t)),$$

and so

$$\begin{aligned} \dot{\mathbf{a}}(t) &= [A - \sum_{\mathsf{j}=\mathsf{i},\mathsf{e}} C_{\mathsf{j}}^{\dagger}S_{\mathsf{ji}}(\eta - S_{\mathsf{ii}})^{-1}C_{\mathsf{i}}]\mathbf{a}(t) \\ &\quad - \sum_{\mathsf{j}=\mathsf{i},\mathsf{e}} C_{\mathsf{j}}^{\dagger}\left(S_{\mathsf{je}} + S_{\mathsf{ji}}(\eta - S_{\mathsf{ii}})^{-1}S_{\mathsf{ie}}\right)\mathbf{b}_{\mathsf{e}}^{\text{in}}(t) \end{aligned}$$

or

$$\begin{aligned} \hat{\mathbf{a}}(s) &= \frac{-1}{s - A_{\text{red}}}\sum_{\mathsf{j}=\mathsf{i},\mathsf{e}} C_{\mathsf{j}}^{\dagger}\left(S_{\mathsf{je}} + S_{\mathsf{ji}}\frac{1}{\eta - S_{\mathsf{ii}}}S_{\mathsf{ie}}\right)\hat{\mathbf{b}}_{\mathsf{e}}(s) \\ &\quad + \frac{1}{s - A_{\text{red}}}\mathbf{a}, \end{aligned}$$

with A_{red} as above. Consequently,

$$\begin{aligned} \hat{\mathbf{b}}_{\mathsf{e}}^{\text{out}} &= S_{\mathsf{ei}}\hat{\mathbf{b}}_{\mathsf{i}}^{\text{in}} + S_{\mathsf{ee}}\hat{\mathbf{b}}_{\mathsf{e}}^{\text{in}} + C_{\mathsf{e}}\hat{\mathbf{a}} \\ &= S_{\text{red}}\hat{\mathbf{b}}_{\mathsf{e}}^{\text{in}} + C_{\text{red}}\hat{\mathbf{a}} \\ &= \Xi_{\text{red}}\hat{\mathbf{b}}_{\mathsf{e}}^{\text{in}} + \xi_{\text{red}}\mathbf{a} \end{aligned}$$

where

$$\begin{aligned} \Xi_{\text{red}} &= S_{\text{red}} - \sum_{\mathsf{j}=\mathsf{i},\mathsf{e}} C_{\text{red}}\frac{1}{s - A_{\text{red}}}C_{\mathsf{j}}^{\dagger}(S_{\mathsf{je}} + S_{\mathsf{ji}}\frac{1}{\eta - S_{\mathsf{ii}}}S_{\mathsf{ie}}), \\ \xi_{\text{red}} &= C_{\text{red}}\frac{1}{s - A_{\text{red}}}, \end{aligned}$$

and S_{red}, C_{red} are as in the statement of the theorem.

We now show that $\sum_{j=i,e} C_j^\dagger [S_{je} + S_{ji} (\eta - S_{ii})^{-1} S_{ie}] = C_{red}^\dagger S_{red}$. Now

$$\begin{aligned} &\sum_{j=i,e} C_j^\dagger [S_{je} + S_{ji} (\eta - S_{ii})^{-1} S_{ie}] \\ &= C_i^\dagger [S_{ie} + S_{ii} (\eta - S_{ii})^{-1} S_{ie}] + C_e^\dagger S_{red} \\ &= C_i^\dagger \eta (\eta - S_{ii})^{-1} S_{ie} + C_e^\dagger S_{red}, \end{aligned}$$

while $C_{red}^\dagger S_{red} = C_i^\dagger (\eta^\dagger - S_{ii}^\dagger)^{-1} S_{ie}^\dagger S_{red} + C_e^\dagger S_{red}$. However,

$$(\eta^\dagger - S_{ii}^\dagger)^{-1} S_{ie}^\dagger S_{red} = (\eta^\dagger - S_{ii}^\dagger)^{-1} S_{ei}^\dagger (S_{ee} + S_{ei} (\eta - S_{ii})^{-1} S_{ie})$$

and using the identities $S_{ii}^\dagger S_{ii} + S_{ei}^\dagger S_{ei} = 1$, $S_{ii}^\dagger S_{ie} + S_{ei}^\dagger S_{ee} = 0$, this reduces to

$$\begin{aligned} &(\eta^\dagger - S_{ii}^\dagger)^{-1} S_{ie}^\dagger S_{red} \\ &= (\eta^\dagger - S_{ii}^\dagger)^{-1} \left[-S_{ii}^\dagger S_{ie} + (1 - S_{ii}^\dagger S_{ii}) (\eta - S_{ii})^{-1} S_{ie} \right] \\ &= (\eta^\dagger - S_{ii}^\dagger)^{-1} \left[-S_{ii}^\dagger (\eta - S_{ii}) + (1 - S_{ii}^\dagger S_{ii}) \right] \frac{1}{\eta - S_{ii}} S_{ie} \\ &= \eta (\eta - S_{ii})^{-1} S_{ie}. \end{aligned}$$

Therefore $\Xi_{red} = S_{red} - \sum_{j=i,e} C_{red} \frac{1}{s - A_{red}} C_{red}^\dagger S_{red}$, as required.

For consistency, we should check that we have $A_{red} = -\frac{1}{e} C_{red}^\dagger C_{red} - i\Omega_{red}$ with Ω_{red} selfadjoint. Indeed, setting $A = -\frac{1}{2} C_i^\dagger C_i - \frac{1}{2} C_e^\dagger C_e - i\Omega$ and substituting in for C_{red} and A_{red} we find after some algebra that

$$\begin{aligned} \Omega_{red} &= \Omega + \mathrm{Im} \left\{ C_i^\dagger S_{ii} (\eta - S_{ii})^{-1} C_i \right\} \\ &\quad + \mathrm{Im} \left\{ C_e^\dagger S_{ei} (\eta - S_{ii})^{-1} C_i \right\}. \end{aligned}$$

The manipulation for this is trivial except for the calculation of the term of the form $\frac{1}{2} C_i^\dagger X C_i$ where

$$\begin{aligned} X &= 1 + 2S_{ii} (\eta - S_{ii})^{-1} - (\eta^\dagger - S_{ii}^\dagger)^{-1} S_{ei}^\dagger S_{ei} (\eta - S_{ii})^{-1} \\ &\equiv (1 - \eta S_{ii}^\dagger)^{-1} \left[S_{ii} \eta^\dagger - \eta S_{ii}^\dagger \right] (1 - S_{ii} \eta^\dagger)^{-1} \\ &= 2i \mathrm{Im} \frac{S_{ii} \eta^\dagger}{1 - S_{ii} \eta^\dagger} = 2i \mathrm{Im} \left\{ S_{ii} (\eta - S_{ii})^{-1} \right\} \end{aligned}$$

where again we use the identity $S_{ii}^\dagger S_{ii} + S_{ei}^\dagger S_{ei} = 1$. ■

In terms of the parameters (S, L, H) with $S = \begin{pmatrix} S_{ii} & S_{ie} \\ S_{ei} & S_{ee} \end{pmatrix}$, $L = \begin{pmatrix} L_i \\ L_e \end{pmatrix} = \begin{pmatrix} C_i \mathbf{a} \\ C_e \mathbf{a} \end{pmatrix}$ and $H = \mathbf{a}^\dagger \Omega \mathbf{a}$, we have that the feedback system is described by the reduced parameters $(S_{red}, L_{red}, H_{red})$ where

$$\begin{aligned} S_{red} &= S_{ee} + S_{ei} (\eta - S_{ii})^{-1} S_{ie} \\ L_{red} &= S_{ei} (\eta - S_{ii})^{-1} L_i + L_e, \\ H_{red} &= H + \mathrm{Im} \left\{ L_i^\dagger S_{ii} (\eta - S_{ii})^{-1} L_i \right\} \\ &\quad + \mathrm{Im} \left\{ L_e^\dagger S_{ei} (\eta - S_{ii})^{-1} L_i \right\}. \end{aligned} \quad (13)$$

The same equations have been deduced in the nonlinear case by different arguments [5]. Note the identity $\mathrm{Im} \left\{ L_i^\dagger S_{ii} (\eta - S_{ii})^{-1} L_i \right\} = \mathrm{Im} \left\{ L_i^\dagger (\eta - S_{ii})^{-1} L_i \right\}$.

Let U be a unitary operator on a fixed Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ which decomposes as $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. The non-commutative Möbius transform $\varphi_U^{2 \to 1}$ is the superoperator. defined by

$$\varphi_U^{2 \to 1} (X) = U_{11} + U_{12} (1 - X U_{22})^{-1} X U_{21}$$

defined on the domain of operators X on $\mathfrak{H}_2$ for which the inverse $(1 - X U_{22})^{-1}$ exists. The transform $\varphi_U^{2 \to 1}$ maps unitaries on $\mathfrak{H}_2$ in its domain to unitaries in $\mathfrak{H}_1$ [17].

In particular, S_{red} is unitary as it equals $\varphi_S^{i \to e} (\xi)$ where $\xi = \eta^{-1}$ with η being unitary. We may expand the geometric series to write

$$S_{red} = S_{ee} + \sum_{n=0}^\infty S_{ei} \xi (S_{ii} \xi)^n S_{ie}$$

which shows that S_{red} can be built up from contributions from the various paths through the network. Likewise

$$\begin{aligned} L_{red} &= L_e + \sum_{n=0}^\infty S_{ei} \xi (S_{ii} \xi)^n L_i, \\ H_{red} &= H + \sum_{n=0}^\infty \mathrm{Im} \left\{ L_i^\dagger (S_{ii} \xi)^n L_i \right\} \\ &\quad + \sum_{n=0}^\infty \mathrm{Im} \left\{ L_e^\dagger S_{ei} \xi (S_{ii} \xi)^n L_i \right\}. \end{aligned}$$

IV. SYSTEMS IN SERIES

As a very special case of feedback connections we consider the situation of systems in series. This is referred to as *feedforward* in engineering.

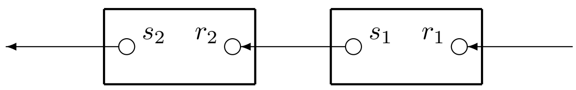

figure 3: Cascaded systems

The individual transfer functions before the connection $e = (s_1, r_2)$ is made are given by $\Xi_i = \left[\begin{array}{c|c} A_i & -C_i^\dagger S_i \\ \hline C_i & S_i \end{array} \right]$ with $A_i = -\frac{1}{2} C_i^\dagger C_i - i\Omega_i$.and these may be concatenated to give

$$\Xi = \left[\begin{array}{c|cc} A_1 + A_2 & -C_1^\dagger S_1 & -C_2^\dagger S_2 \\ \hline C_1 & S_1 & 0 \\ C_2 & 0 & S_2 \end{array} \right].$$

To use the formula for the reduced transfer function following connection, we must first of all identify the internal (eliminated) and external fields: here

$$\mathbf{b}^{\mathrm{in}} = \begin{pmatrix} \mathbf{b}^{\mathrm{in}}_{\mathsf{i}} \\ \mathbf{b}^{\mathrm{in}}_{\mathsf{e}} \end{pmatrix} = \begin{pmatrix} \mathbf{b}^{\mathrm{in}}_{2} \\ \mathbf{b}^{\mathrm{in}}_{1} \end{pmatrix}, \mathbf{b}^{\mathrm{out}} = \begin{pmatrix} \mathbf{b}^{\mathrm{out}}_{\mathsf{i}} \\ \mathbf{b}^{\mathrm{out}}_{\mathsf{e}} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{b}^{\mathrm{out}}_{1} \\ \mathbf{b}^{\mathrm{out}}_{2} \end{pmatrix},$$

and

$$\begin{pmatrix} S_{\mathsf{ii}} & S_{\mathsf{ie}} \\ S_{\mathsf{ei}} & S_{\mathsf{ee}} \end{pmatrix} \equiv \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}, \quad L_{\mathsf{i}} \equiv L_1, L_{\mathsf{e}} \equiv L_2,$$

with trivially $\eta = 1$. The reduced transfer function is then readily computed to be

$$\Xi_{\mathrm{series}} = \left[\begin{array}{c|c} A_1 + A_2 - C_2^\dagger S_2 C_1 & -\left(C_2^\dagger S_2 + C_1^\dagger \right) S_1 \\ \hline C_2 + S_2 C_1 & S_2 S_1 \end{array} \right].$$

Likewise we deduce the relations

$$S = S_2 S_1,\ L = L_2 + S_2 L_1,\ H = H_1 + H_2 + \mathrm{Im}\left\{ L_2^\dagger S_2 L_1 \right\}. \tag{14}$$

The same equations have been deduced in the nonlinear case by different arguments [6].

A. Feedforward: Cascades

If the two systems are truly distinct systems, that is, if they are different sets of oscillators, then we are in the situation of properly *cascaded* systems [16]. In this case one would expect that the transfer function to factor as the ordinary matrix product $\Xi_{\mathrm{series}} \equiv \Xi_2 \Xi_1$. We now show that this is indeed the case.

Lemma: Let Ξ_j be transfer functions for m_j oscillators coupled to n fields ($j = 1, 2$). If we consider the ampliated transfer functions for $m_1 + m_2$ oscillators coupled to n fields

$$\begin{aligned} \tilde{\Xi}_1 &= \left[\begin{array}{c|c} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -C_1^\dagger S_1 \\ 0 \end{pmatrix} \\ \hline (C_1, 0) & S_1 \end{array} \right], \\ \tilde{\Xi}_2 &= \left[\begin{array}{c|c} \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix} & \begin{pmatrix} 0 \\ -C_2^\dagger S_2 \end{pmatrix} \\ \hline (0, C_2) & S_2 \end{array} \right], \end{aligned}$$

then

$$\tilde{\Xi}_{\mathrm{series}} = \Xi_2 \Xi_1. \tag{15}$$

Proof: We compute this directly,

$$\tilde{\Xi}_{\mathrm{series}} = \left[\begin{array}{c|c} \begin{pmatrix} A_1 & 0 \\ -C_2^\dagger S_2 C_1 & -A_2 \end{pmatrix} & \begin{pmatrix} -C_1^\dagger S_1 \\ -C_2^\dagger S_2 S_1 \end{pmatrix} \\ \hline (C_1, C_2) & S_2 S_1 \end{array} \right]$$

$$= S_2 S_1 - (C_1, C_2) \begin{pmatrix} s - A_1 & 0 \\ C_2^\dagger S_2 C_1 & s - A_2 \end{pmatrix}^{-1} \begin{pmatrix} C_1^\dagger S_1 \\ -C_2^\dagger S_2 S_1 \end{pmatrix},$$

with the inverse matrix equal to

$$\begin{pmatrix} \frac{1}{s - A_1} & 0 \\ -\frac{1}{s - A_2} C_2^\dagger S_2 C_1 \frac{1}{s - A_1} & \frac{1}{s - A_2} \end{pmatrix}$$

and expanding out gives the result

$$\tilde{\Xi}_{\mathrm{series}} = \prod_{\alpha = 1,2} \left[S_\alpha - C_\alpha \left(s - A_\alpha \right)^{-1} C_\alpha^\dagger S_\alpha \right].$$

■

V. BEAM SPLITTERS

A simple beam splitter is a device performing physical superposition of two input fields. It is described by a fixed unitary operator $T = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix} \in U(2)$:

$$\begin{pmatrix} \mathbf{b}_1^{\mathrm{out}} \\ \mathbf{b}_2^{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix} \begin{pmatrix} \mathbf{b}_1^{\mathrm{in}} \\ \mathbf{b}_2^{\mathrm{in}} \end{pmatrix}.$$

This is a canonical transformation and the output fields satisfy the same canonical commutation relations as the inputs. The action of the beam splitter is depicted in the figure below. On the left we have a traditional view of the two inputs being split into two output fields. On the right we have our view of the beam splitter as being a component with two input ports and two output ports: we have sketched some internal detail to emphasize the scattering (superimposing) of inputs however we shall usually just draw this as a "black box" component in the following.

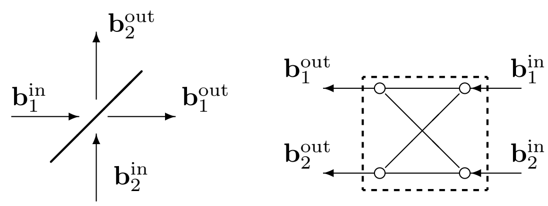

Figure 4: Beam-splitter component.

To emphasize that the beam splitter is an input-output device of exactly the form we have been considering up to now, let us state that its transfer matrix function is

$$\Xi_{\text{beam splitter}} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & T \end{array} \right] \equiv T.$$

Our aim is to describe the effective Markov model for the feedback device sketched below where the feedback is implemented by means of a beam splitter. Here we have a component system, called the plant, in-loop and we assume that it is described by the transfer function $\Xi_0 = \left[\begin{array}{c|c} A_0 & -C_0^\dagger S_0 \\ \hline C_0 & S_0 \end{array} \right]$.

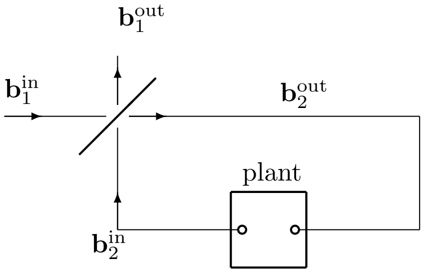

Figure 5: Feedback using a beam-splitter.

It is more convenient to view this as the network sketched below.

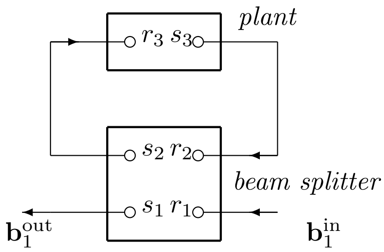

Figure 6: Network representation.

Here we have the pair of internal edges (s_2, r_3) and (s_3, r_2). The transfer function for the network is

$$\Xi_{\rm unconn.} = \left[\begin{array}{c|ccc} A_0 & 0 & 0 & -C_0^\dagger S_0 \\ \hline 0 & T_{11} & T_{12} & 0 \\ 0 & T_{21} & T_{22} & 0 \\ C_0 & 0 & 0 & S_0 \end{array}\right]$$

with respect to the labels $(0, s_1, s_2, s_3)$ for the rows and $(0, r_1, r_2, r_3)$ for the columns. This time the external fields are $\mathbf{b}_{\sf e}^{\rm in} = \mathbf{b}_1^{\rm in}$, $\mathbf{b}_{\sf e}^{\rm out} = \mathbf{b}_1^{\rm out} \equiv T_{11}\mathbf{b}_1^{\rm in} + T_{12}\mathbf{b}_2^{\rm in}$ while the (matched) internal fields are

$$\mathbf{b}_{\sf i}^{\rm in} = \begin{pmatrix} \mathbf{b}_2^{\rm in} \\ \mathbf{b}_3^{\rm in} \end{pmatrix}, \ \mathbf{b}_{\sf i}^{\rm out} = \begin{pmatrix} \mathbf{b}_2^{\rm out} \\ \mathbf{b}_3^{\rm out} \end{pmatrix} \equiv \begin{pmatrix} T_{21}\mathbf{b}_1^{\rm in} + T_{22}\mathbf{b}_2^{\rm in} \\ S_0 \mathbf{b}_3^{\rm in} + L_0 \end{pmatrix}.$$

That is

$$\begin{aligned} S_{\sf ii} &= \begin{pmatrix} T_{22} & 0 \\ 0 & S_0 \end{pmatrix}, & S_{\sf ie} &= \begin{pmatrix} T_{21} \\ 0 \end{pmatrix}, \\ S_{\sf ei} &= (T_{12}, 0), & S_{\sf ee} &= T_{11}, \\ L_{\sf i} &= \begin{pmatrix} 0 \\ L_0 \end{pmatrix}, & L_{\sf e} &= 0, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Substituting into our reduction formula we obtain

$$\begin{aligned} S &= T_{11} + (T_{12}, 0) \begin{pmatrix} -T_{22} & 1 \\ 1 & -S_0 \end{pmatrix}^{-1} \begin{pmatrix} T_{21} \\ 0 \end{pmatrix} \\ &\equiv T_{11} + T_{12} (1 - S_0 T_{22})^{-1} S_0 T_{21}, \\ C &= (T_{12}, 0) \begin{pmatrix} -T_{22} & 1 \\ 1 & -S_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C_0 \end{pmatrix} \\ &\equiv T_{12} (1 - S_0 T_{22})^{-1} C_0, \\ \Omega &= \Omega_0 + {\rm Im}\, (0, C_0^\dagger) \begin{pmatrix} -T_{22} & 1 \\ 1 & -S_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C_0 \end{pmatrix} \\ &\equiv \Omega_0 + {\rm Im} C_0^\dagger (1 - S_0 T_{22})^{-1} C_0. \end{aligned}$$

and so, when the connections are made, the transfer matrix function is

$$\left[\begin{array}{c|c} A_0 - C_0^\dagger S_0 T_{22} C_0 & -C_0^\dagger S_0 T_{21} - C_0^\dagger S_0 \frac{1}{1 - S_0 T_{22}} T_{22} \\ \hline T_{12} \frac{1}{1 - S_0 T_{22}} C_0 & T_{11} + T_{12} \frac{1}{1 - S_0 T_{22}} S_0 T_{21} \end{array}\right],$$

Note that $S = \varphi_T^{2\to 1}(S_0)$ where $\varphi_T^{2\to 1}(z) = T_{11} + T_{12}\beta\left(z^{-1} - T_{22}\right) T_{21}$ is the Möbius transformation in the complex plane associated with T.

If we further set $T = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix}$, and $x + iy = S_0 \nu$, then

$$\begin{aligned} C^\dagger C &= \left| \frac{\beta}{1 - S_0 \nu} \right|^2 C_0^\dagger C_0 = \frac{1 - |\nu|^2}{|1 - S_0 \nu|^2} C_0^\dagger C_0 \\ &\equiv \frac{1 - x^2 - y^2}{(1 - x)^2 + y^2} C_0^\dagger C_0, \\ {\rm Im} C_0^\dagger (1 - S_0 \nu)^{-1} C_0 &= {\rm Im} \left\{ \frac{1}{1 - x - iy} \right\} C_0^\dagger C_0 \\ &\equiv \frac{y}{(1 - x)^2 + y^2} C_0^\dagger C_0. \end{aligned}$$

In particular, if we take a single oscillator in-loop with $S_0 = e^{i\phi_0}$, then we obtain $S \equiv e^{i\phi}$ and the phase is determined by the Möbius transformation. If we further have $L_0 = \sqrt{\gamma_0} a$, $H_0 = \omega_0 a^\dagger a$, we find that $L \equiv e^{i\delta}\sqrt{\gamma} a$ and $H = \omega a^\dagger a$ where

$$\gamma = \frac{1 - x^2 - y^2}{(1 - x)^2 + y^2} \gamma_0, \quad \omega = \frac{y}{(1 - x)^2 + y^2} \omega_0,$$

and δ is a real phase. In the specific case $T = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$ with $S_0 = 1, \omega_0 = 0$ considered by Yanagisawa and Kimura [3], we have $x = -\alpha$ and $y = 0$, therefore we find

$$\gamma = \frac{1 - \alpha}{1 + \alpha} \gamma_0, \quad \omega = 0$$

which agrees with their findings.

An alternative computation of Ξ is given by the following argument. We consider the input-output relations

$$\hat{\mathbf{b}}_i^{\rm out} = \sum_{j=1,2} T_{ij} \hat{\mathbf{b}}_j^{\rm in}, \quad \hat{\mathbf{b}}_2^{\rm in} = \Xi_0 \hat{\mathbf{b}}_1^{\rm out} + \xi_0 \mathbf{a}_0,$$

and eliminating $\hat{\mathbf{b}}_2^{\text{out}} \equiv (1 - T_{22}\Xi)^{-1}\left[T_{21}\hat{\mathbf{b}}_1^{\text{in}} + T_{22}\xi_0\mathbf{a}_0\right]$ yields

$$\begin{aligned}\hat{\mathbf{b}}_1^{\text{out}} &= \left[T_{11} + T_{12}\Xi_0\left(1 - T_{22}\Xi_0\right)^{-1} T_{21}\right]\hat{\mathbf{b}}_1^{\text{in}} \\ &\quad + T_{21}\left(1 - \Xi_0 T_{22}\right)^{-1}\xi_0\mathbf{a}_0.\end{aligned}$$

That is

$$\Xi = T_{11} + T_{12}\left(\Xi_0^{-1} - T_{22}\right)^{-1} T_{21} = \varphi_T^{2\to 1}\left(\Xi_0\right).$$

We remark that if T_{12} and T_{21} are invertible, then we may invert the Möbius transformation to get

$$\Xi_0^{-1} = T_{22} + T_{21}\frac{1}{\Xi - T_{11}}T_{12}.$$

To illustrate with a cavity mode in-loop, we take the beam splitter matrix to be $T = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$ with $\alpha^2 + \beta^2 = 1$, and the transfer function $\Xi_0(s) = \frac{s + i\omega - \gamma/2}{s + i\omega + \gamma/2}$, then we find

$$\Xi = \frac{\alpha + \Xi_0}{1 + \alpha\Xi_0} = \frac{s + i\omega - \frac{1-\alpha}{1+\alpha}\frac{\gamma}{2}}{s + i\omega + \frac{1-\alpha}{1+\alpha}\frac{\gamma}{2}}.$$

VI. THE REDHEFFER STAR PRODUCT

An important feedback arrangement is shown in the figure below.

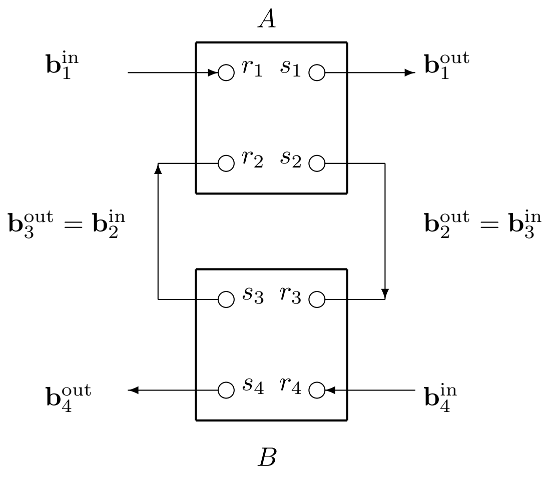

Figure 7: Composite System

We shall now derive the matrices for this system taking component A to be described $\begin{pmatrix} S_{11}^A & S_{12}^A \\ S_{21}^A & S_{22}^A \end{pmatrix}$, $\begin{pmatrix} C_1^A \\ C_2^A \end{pmatrix}$, Ω_A and B by $\begin{pmatrix} S_{33}^B & S_{34}^B \\ S_{43}^B & S_{44}^B \end{pmatrix}$, $\begin{pmatrix} C_3^B \\ C_4^B \end{pmatrix}$, Ω_B. The operators of systems A are assumed to commute with those of B. We have two internal channels to eliminate which we can do in sequence, or simultaneously. We shall do the latter. here we have

$$\begin{aligned}S_{\mathtt{ee}} &= \begin{pmatrix} S_{11}^A & 0 \\ 0 & S_{44} \end{pmatrix}, S_{\mathtt{ei}} = \begin{pmatrix} S_{12}^A & 0 \\ 0 & S_{43}^B \end{pmatrix} \\ S_{\mathtt{ie}} &= \begin{pmatrix} S_{21}^A & 0 \\ 0 & S_{34}^B \end{pmatrix}, S_{\mathtt{ii}} = \begin{pmatrix} S_{22}^A & 0 \\ 0 & S_{33}^B \end{pmatrix}\end{aligned}$$

and

$$L_{\mathtt{e}} = \begin{pmatrix} L_1^A \\ L_4^B \end{pmatrix}, \quad L_{\mathtt{i}} = \begin{pmatrix} L_2^A \\ L_3^B \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The parameters are therefore

$$\begin{aligned}S_\star = &\begin{pmatrix} S_{11}^A & 0 \\ 0 & S_{44}^B \end{pmatrix} \\ &+ \begin{pmatrix} S_{12}^A & 0 \\ 0 & S_{43}^B \end{pmatrix}\begin{pmatrix} -S_{22}^A & 1 \\ 1 & -S_{33}^B \end{pmatrix}^{-1}\begin{pmatrix} S_{21}^A & 0 \\ 0 & S_{34}^B \end{pmatrix},\end{aligned}$$

$$L_\star = \begin{pmatrix} L_1^A \\ L_4^B \end{pmatrix} + \begin{pmatrix} S_{12}^A & 0 \\ 0 & S_{43}^B \end{pmatrix}\begin{pmatrix} -S_{22}^A & 1 \\ 1 & -S_{33}^B \end{pmatrix}^{-1}\begin{pmatrix} L_2^A \\ L_3^B \end{pmatrix}.$$

That is,

$$\begin{aligned}S_{\star,11} &= S_{11}^A + S_{12}^A S_{33}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} S_{21}^A, \\ S_{\star,14} &= S_{12}^A\left(1 - S_{22}^A S_{33}^B\right)^{-1} S_{34}^B, \\ S_{\star,41} &= S_{43}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} S_{21}^A, \\ S_{\star,44} &= S_{44} + S_{43}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} S_{22}^A S_{34}^B.\end{aligned}$$

While letting $\mathbf{a}_A$ and $\mathbf{a}_B$ be the modes in systems A and B respectively,

$$\begin{aligned}L_{\star,1} &= \left\{C_1^A + S_{12}^A S_{33}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} C_2^A\right\}\mathbf{a}_A \\ &\quad + S_{12}^A\left(1 - S_{22}^A S_{33}^B\right)^{-1} C_3^B\mathbf{a}_B, \\ L_{\star,4} &= S_{43}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} C_2^A\mathbf{a}_A \\ &\quad + \left\{C_4^B + S_{43}^B S_{22}^A\left(1 - S_{22}^A S_{33}^B\right)^{-1} C_3^B\right\}\mathbf{a}_B,\end{aligned}$$

and

$$H_\star = \mathbf{a}_A^\dagger\left(\Omega_A + \Lambda_A\right)\mathbf{a}_A + \mathbf{a}_B^\dagger\left(\Omega_B + \Lambda_B\right)\mathbf{a}_B + \mathbf{a}_B^\dagger\Lambda_{BA}\mathbf{a}_B,$$

where

$$\begin{aligned}\Lambda_A &= \operatorname{Im}\left\{\begin{array}{c} C_2^{A\dagger}\left(1 - S_{22}^A S_{33}^B\right)^{-1} C_2^A \\ + C_1^{A\dagger} S_{12}^A\left(1 - S_{33}^B S_{22}^A\right)^{-1} S_{33}^B C_2^A \end{array}\right\}, \\ \Lambda_B &= \operatorname{Im}\left\{\begin{array}{c} C_3^{B\dagger}\left(1 - S_{33}^B S_{22}^A\right)^{-1} C_3^B \\ + C_4^{B\dagger} S_{43}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} S_{22}^A C_3^B \end{array}\right\}, \\ \Lambda_{AB} &= \operatorname{Im}\left\{\begin{array}{c} C_3^{B\dagger}\left(1 - S_{33}^B S_{22}^A\right)^{-1} S_{33}^B C_2^A \\ + C_4^{B\dagger} S_{43}^B\left(1 - S_{22}^A S_{33}^B\right)^{-1} C_2^A \\ - C_2^{A\dagger}\left(1 - S_{22}^A S_{33}^B\right)^{-1} S_{22}^A C_3^B \\ - C_1^{A\dagger} S_{12}^A\left(1 - S_{33}^B S_{22}^A\right)^{-1} C_3^B \end{array}\right\}.\end{aligned}$$

VII. APPENDIX

A. Quantum Itō Calculus

It is convenient to introduce integrated fields

$$B_i(t) \equiv \int_0^t b_i(s)\,ds, B_i^\dagger(t) \equiv \int_0^t b_i^\dagger(s)\,ds,$$
$$\Lambda_{ij}(t) \equiv \int_0^t b_i^\dagger(s)\,b_j(s)\,ds.$$

$B_i(t)$ and $B_i^\dagger(t)$ are called the annihilation and creation process, respectively, for the ith field and collectively are referred to as a quantum Wiener process. $\Lambda_{ij}(t)$ is called the gauge process or scattering process from the jth field to the ith field. A noncommutative version of the Ito theory of stochastic integration with respect to these processes can be built up. The quantum Itō table giving the product of infinitesimal increments of these process is

$\times$	dB_k	$d\Lambda_{kl}$	$dB_l^\dagger$	dt
dB_i	0	$\delta_{ik}dB_l$	$\delta_{il}dt$	0
$d\Lambda_{ij}$	0	$\delta_{jk}d\Lambda_{il}$	$\delta_{jl}dB_i^\dagger$	0
$dB_j^\dagger$	0	0	0	0
dt	0	0	0	0

,

and we note the quantum Itō product rule for adapted stochastic integral processes [11]

$$d(XY) = (dX)\,Y + X\,(dY) + (dX)\,(dY).$$

The Itō equation for the unitary process is then $dV = (dG)\,V$ where

$$dG = \sum_{i,j=1}^{n} (S_{ij} - \delta_{ij}) \otimes d\Lambda_{ij} + \sum_{i=1}^{n} L_i \otimes dB_i^\dagger - \sum_{j=1}^{n} L_i^\dagger S_{ij} \otimes dB_j - (\frac{1}{2}\sum_{i=1}^{n} L_i^\dagger L_i - iH) \otimes dt.$$

B. Dynamical Equations

Let X_t be a quantum stochastic integral, then the product rule yields

$$d(V_t^\dagger X_t V_t) = (dV_t^\dagger)X_tV_t + V_t^\dagger\,(dX_t)\,V_t + V_t^\dagger X_t(dV_t)$$
$$+(dV_t^\dagger)(dX_t)V_t + (dV_t^\dagger)X_t(dV_t) + V_t^\dagger(dX_t)(dV_t) +(dV_t^\dagger)(dX_t)(dV_t).$$

In the case where $X_t = X \otimes 1 \in B(\mathfrak{h} \otimes \mathcal{E})$, a constant operator on the system space $\mathfrak{h}$, we have $dX = 0$ so

$$d\left(V_t^\dagger X_t V_t\right) = \left(dV_t^\dagger\right) XV_t + V_t^\dagger X(dV_t) + (dV_t^\dagger)X(dV_t)$$
$$= V_t^\dagger\left(S_{ki}^\dagger X S_{kj} - \delta_{ij}X\right) V_t d\Lambda_{jk} +V_t^\dagger S_{ki}^\dagger\,[X, L_k]\,V_t dB_i^\dagger + V_t^\dagger[L_i^\dagger, X]S_{ij}V_t dB_j +V_t^\dagger\left\{\frac{1}{2}L_k^\dagger\,[X, L_k] + \frac{1}{2}[L_k^\dagger, X]L_k - i\,[X, H]\right\} V_t,$$

This is clearly the Heisenberg-Langevin equation from section II.

The output fields are given by $B_i^{\rm out}(t) = V_t^\dagger B_i(t)\,V_t$. We also note that

$$d\left(V_t^\dagger B_i(t)\,V_t\right) = dB_i(t) + V_t^\dagger(dB_i(t))(dV_t))$$

with all other terms cancelling on the right hand side, leaving

$$dB_i(t) + V_t^\dagger(S_{ij} - \delta_{ij})V_t dB_j(t) + V_t^\dagger L_i V_t dt = V_t^\dagger S_{ij} V_t dB_j(t) + V_t^\dagger L_i V_t dt,$$

which is the desired input-output relation.

C. Stratonovich to Itō Conversion

The Stratonovich form is $dV = -i\,(dE) \circ V$ where

$$dE = \sum_{i,j=1}^{n} E_{ij} d\Lambda_{ij} + \sum_{i=1}^{n} F_i dB_i^\dagger + \sum_{j=1}^{n} F_j^\dagger dB_j(t) + K dt.$$

and we define the Stratonovich differential to be $(dX) \circ Y = (dX)\,Y + \frac{1}{2}\,(dX)\,(dY)$ with the last term computed using the Itō table. We have the consistency condition $dV = (dG)\,V \equiv -i\,(dE)\,V - \frac{i}{2}\,(dE)\,(dG)\,V$ or

$$dG = -idE - \frac{i}{2}\,(dE)\,(dG),$$

and using the table we see that

$$S - 1 \equiv -iE - \frac{i}{2}E\,(S-1)$$
$$L = -iF - \frac{i}{2}EL$$
$$-\frac{1}{2}L^\dagger L - iH = -iK - \frac{i}{2}F^\dagger L$$

which can be solved to give the relations (3).

D. Analytic Properties of the Transfer Functions

In particular in the stable case (i.e., C invertible) the transfer function Ξ is analytic on the closed right half plane and unitary on the whole imaginary axis. In other words, Ξ is a (matrix-valued rational) inner function.

Whenever appropriate, we may determine Ξ from its (unitary) values on the imaginary axis by using the

Poisson integral formula (see Chapter 8 in [15]): For $x > 0, y \in \mathbb{R}$

$$\Xi\left(x+iy\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \Xi\left(i\omega\right) \frac{x}{x^2 + (y-\omega)^2}\, d\omega.$$

Note that Ξ is also analytic at infinity (with value S), hence if convenient we can always Cayley-transform it to an inner function on the unit disc. Some immediate consequences:

(a) $\|\Xi(s)\| \leq 1$ if $Re(s) \geq 0$ (operator norm, maximum principle)

(b) Ξ maps the (vector-valued) Hardy spaces H^p into themselves.

(c) If $n = 1$ (one input and one output field) then Ξ is a scalar rational inner function. It is known that all such functions are finite Blaschke products [15], i.e., finite products of the special form Ξ_{cavity} in the example above. In other words, such a system can always be realized by finitely many systems of this special form in series (compare Section 4).

The relevance of Hardy spaces and inner functions in classical control theory is discussed in [17]. An introduction to operator-valued inner functions is given in [18].

In general, the real and imaginary parts of A need not commute - that is, the commutator $\left[C^{\dagger}C, \Omega\right]$ need not be identically zero. However, when this does occur we always recover a multi-mode version of the cavity situation.

---

[1] C. Gardiner and P. Zoller, Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics, 2nd ed., ser. Springer Series in Synergetics. Springer, (2000).

[2] H.Wiseman, Phys. Rev. A, vol. 49, no. 3, 2133-2150, (1994)

[3] M. Yanagisawa, H. Kimura, *Transfer function approach to quantum control Part I: Dynamics of quantum feedback systems*, IEEE Transactions on Automatic Control, **48**, No. 12, 2107-2120, December (2003)

[4] M. Yanagisawa, H. Kimura, *Transfer function approach to quantum control Part II: Control concepts and applications*, IEEE Transactions on Automatic Control, **48**, No. 12, 2121-2132, December (2003)

[5] J. Gough, M.R. James, *Quantum Feedback Networks: Hamiltonian Formulation*, arXiv:0804.3442(v2) [quant-ph]

[6] J. Gough, M.R. James, *The series product and its application to feedforward and feedback networks*, arXiv:0707.0048(v1) [quant-ph]

[7] M. R. James, H. I. Nurdin, and I. R. Petersen, *H^∞ control of linear quantum stochastic systems*, 2007, accepted in IEEE Transactions on Automatic Control. arXiv:0703150 [quant-ph]

[8] S. Lloyd, *Coherent quantum control*, Phys. Rev. A, 62:022108, (2000)

[9] H. Mabuchi, *Coherent-feedback control with a dynamic compensator*, submitted for publication, preprint: arXiv:0803.2007 [quant-ph].

[10] H. I. Nurdin, M. R. James, and I. R. Petersen, *Quantum LQG control with quantum mechanical controllers*, 2008, to be presented at the 17th IFAC World Congress (Seoul, South Korea, July 6-11, 2008), arXiv:0711.2551

[11] R. L. Hudson and K. R. Parthasarathy, *Quantum Ito's formula and stochastic evolutions,* Commun. Math. Phys. **93**, 301-323 (1984)

[12] K. Parthasarathy, An Introduction to Quantum Stochastic Calculus. Berlin: Birkhauser, 1992.

[13] J. Gough, *Quantum Stratonovich calculus and the quantum Wong-Zakai theorem*, J. Math. Phys., vol. 47, no. 113509, 2006.

[14] P. Halmos, A Hilbert Space Problem Book, 2nd ed., Springer, 1982.

[15] K. Hoffman, Banach Spaces of Analytic Functions, Dover Publications, Reprint, 1988

[16] C.W. Gardiner, Phys. Rev. Lett., **70**:2269-2272, (1993).

[17] N. Young, An Introduction to Hilbert Space, Cambridge Mathematical Textbooks, (1988)

[18] H. Helson, Lectures on Invariant Subspaces, Academic Press, 1964

Making a Neural Network, Quantum. Hello world, we are in Xanadu. Tom Bromley.Â Quantum machine learning is one of the primary focuses at Xanadu. Our machine learning team is strengthening the connections between artificial intelligence and quantum technology. In this blog post we discuss how a neural network can be made quantum, potentially giving huge increases in operating speed and network capacity. This post will require no prior scientific or mathematical background, even if youâ€™ve never heard of a neural network â€” read on! For more details, a paper explaining these findings is available here. Neural networks. You have probably benefited from machine learning today.